SYNTHESIS OF MEASUREMENT SYSTEMS

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SUMMARY

The purpose of this study is to develop a synthesis method for a class of measurement systems. A method has been developed which is applicable to the design of systems with multiple design requirements and constraints, and to the design of systems with both stochastic and deterministic inputs. Application of the method results in the determination of the structure and parameters of the synthesized system.

The synthesis problem is formulated as a problem in the calculus of variations. To do this, it is necessary to first develop a mathematical model for a measurement system. This model describes the dynamic behavior of the specified unalterable system elements, the system inputs, and the system disturbances. It is also necessary to state the various design requirements and constraints mathematically in terms of a system performance index. The synthesis problem is then posed as the variational problem of determining the structure and parameters of the system which is optimum with respect to the performance index, subject to the system dynamic constraints. A dynamic programming approach to the calculus of variations is employed to determine the optimum system.

The system mathematical model is defined, using state variable notation, as a vector differential equation. The state vector associated with this vector differential equation

contains the elements necessary to completely specify the mathematical model.

It is assumed that the specified system dynamic elements are linear so that they are readily represented by linear differential equations. A large class of deterministic input and disturbance signals can also be represented as solutions to appropriate differential equations.

In the more general case where some of the input and disturbance signals are stochastic, it is necessary to incorporate a mathematical model for a random process into the system model. This is accomplished by showing that the first two statistical moments of any stationary or nonstationary random process can be represented as the output of a linear filter excited by white noise. The mathematical model for a random process is then obtained by determining the vector differential equation which defines the linear filter. This differential equation is determined from the first two statistical moments of the given random process.

The system design requirements and constraints are stated mathematically in terms of a system performance index. The performance index is defined as the integral over the measurement interval of a system error functional which is a measure of the instantaneous deviation of the system performance from the ideal. The error functional is different for different system applications and depends largely upon the physical intuition of the system designer. General mathematical restrictions, which must be satisfied by usable error functionals, are determined.

Since the systems considered are in general stochastic, the system performance index can be optimized only in a statistical sense. The synthesis problem then becomes one of determining the system which minimizes the expected value of the performance index, subject to the dynamic constraints imposed by the system mathematical model. Application of the dynamic programming formalism to this problem results in a functional equation which relates the minimum of the expected value of the performance index to the parameters of the optimum system. The limiting form of this functional equation is a partial differential equation. This equation is solved for two important classes of performance indices; a squared error index, and a final value index. In these cases, it is shown that the optimum system is a linear feedback system with time varying feedback gains.

It is necessary to investigate system stability in order to determine the conditions under which the synthesis procedure results in a stable system. This is done by employing methods of Lyapunov stability theory. In particular, it is shown that the synthesis procedure results in a stable system, if the solution to the functional equation is bounded from above, when the measurement interval is infinite. It is also shown that system stability is automatically guaranteed for the cases of minimum squared error and final value performance indices.

Example problems are presented to illustrate the steps involved in the synthesis method.

CHAPTER I

INTRODUCTION

Problem Outline

The problem considered in this study is that of synthesizing a class of measurement systems. A "measurement system" is defined as a system whose function is to produce an output which is an estimate of the state of the system input. Included in this definition are instrumentation systems, servomechanisms, and tracking systems. The design of such systems frequently involves the necessity of accommodating several design requirements and constraints simultaneously. Further, it may be required to measure several inputs which are in general time varying and may be either random or nonrandom. Since it is necessary to provide physical outputs to convey the measurement information, specified dynamic output elements must be included in the measurement system. Inclusion of such elements imposes several physical constraints upon the problem which must be considered in the system design. Additional constraints may enter into the problem in the form of unwanted disturbance signals. As a result of these various constraints, system design by means of conventional analysis techniques becomes impractical. Therefore, synthesis techniques capable of accommodating the imposed constraints are required.

The design of measurement systems, such as those under consideration here, has been considered extensively from an analysis point of view. Conventional system analysis techniques are presented in the books by Truxal (25) and Laning and Battin (26). Hammond (1) presents an approach to measurement system analysis which is useful in formulating synthesis procedures.

Techniques for the synthesis of measurement systems are based on the theory of system optimization. Using this theory, a system performance index is defined and then an effort is made to determine a system which is optimum with respect to the performance index. Such techniques originate in the literature with the basic work of Wiener (2) on optimal filter theory. Using conventional calculus of variation techniques, Wiener showed that optimal filter theory problems lead to the Wiener-Hopf integral equation. Wiener was able to solve this equation and determine the parameters of an optimal filter in the case where the system input and disturbance signals are stationary and the measurement interval is infinite. Bode and Shannon (3) presented a simplification of Wiener's method by developing a model for a stationary random process in the form of a linear time invariant filter excited by white noise. Various extensions of Wiener's method to include systems with nonstationary inputs and finite measurement intervals appear in the literature (4 - 11). The methods presented by these authors involve various techniques for the solution of the Wiener-Hopf integral equation.

Application of optimal filter theory techniques to the design of systems with specified dynamic system constraints is limited due to an inherent difficulty in treating such constraints with the formalism of the classical calculus of variations. Bellman (12) has developed an alternate approach to the calculus of variations which facilitates consideration of these dynamic constraints. This approach, called dynamic programming, has been applied by Bellman to a variety of optimization problems in engineering and economics (12, 13). Merriam (14) has applied the dynamic programming formalism to the design of a class of automatic control systems with deterministic inputs. The dynamic programming formalism, however, has not been extended in any general manner to the optimization of continuous stochastic systems. The work of Levy (15), on the representation of Gaussian random processes, enables the model for a random process presented by Bode and Shannon to be extended to a more general class of random processes in a form which is compatible with the dynamic programming formalism.

The purpose of this study is to present a synthesis method which is applicable to the design of systems with multiple design requirements and constraints, and to systems with both stochastic and deterministic inputs and disturbances. The treatment of stochastic systems is not restricted to stationary random processes, nor to infinite measurement intervals. The method is developed using the dynamic programming techniques. The extension of this technique to include stochastic systems is facilitated by using a model for a random process which is based upon the work of Levy.

Stability criteria, based upon the work of Bertram and Kalman (16), and Bertram and Sarachik (17), are established for the synthesized systems.

Examples which illustrate the synthesis procedure are presented.

CHAPTER II

PROBLEM FORMULATION

In this chapter a general physical description of the problem is given. The general class of systems to be considered is defined, and the necessary assumptions are discussed. The design or synthesis problem is formulated and an outline of the synthesis procedure is given. The last section of the chapter is concerned with the matter of notation.

System Description

The class of systems to be considered in this study can be represented by the block diagram shown in Figure 1. In this figure, the double lines are used to call attention to the fact that the block diagram is an abstract representation which indicates the flow of state variables in the system. Each state variable is a collection of one or more physical quantities. In the case of a single input - single output system, the double lines in Figure 1 could be replaced by single lines representing the single variable physical quantities.

In Figure 1, the input process is to be measured over some finite time interval of duration T. The plant is the output subsystem and consists of specified dynamic elements such as meter movements, servo motors, etc., which are necessary to produce the required physical outputs. The plant is

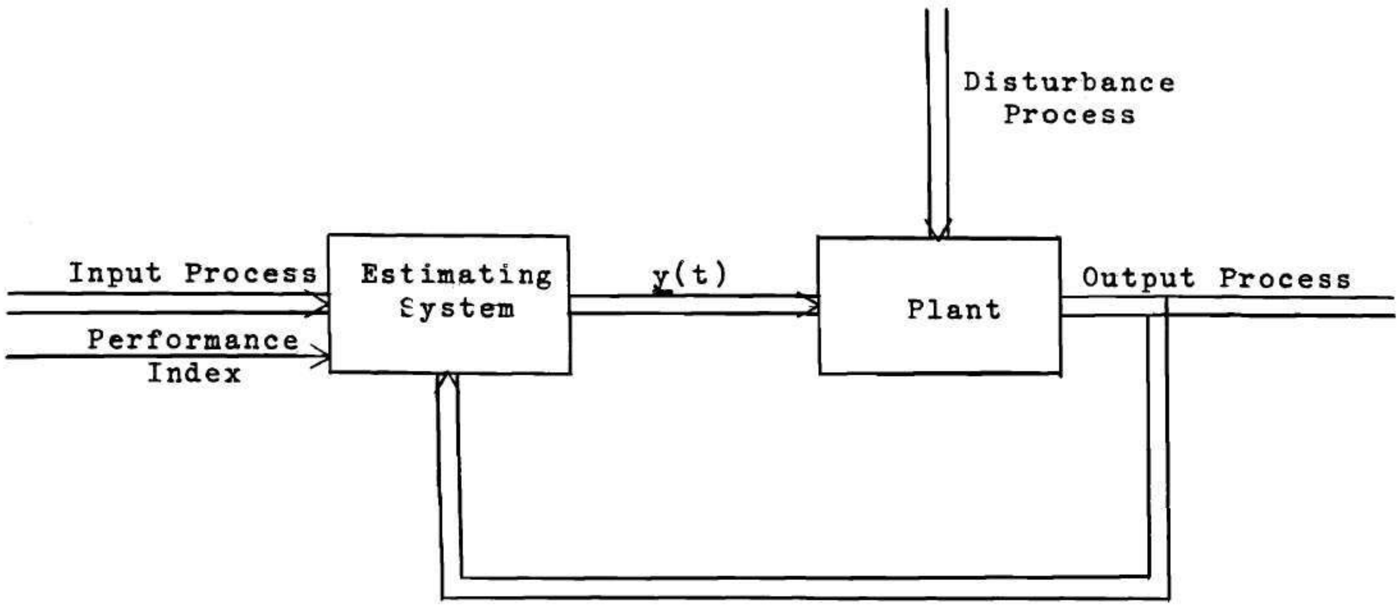

Figure 1. Measurement System Block Diagram

in general subject to disturbance signals which for instance might consist of electrical noise or random load disturbances. The estimating system is an unspecified subsystem to be determined by the design procedure. The estimating system functions as a transducer which accepts the system inputs and produces the signal necessary to actuate the plant. The determination of the estimating system parameters is to be done in such a manner that the over-all system performance is optimum. Optimum system performance is defined in terms of a preassigned system performance index. The performance index for instance might be a measure of the error between a single input variable and a single output variable, or between a collection of inputs and outputs each weighted in a preassigned manner.

The design or synthesis problem associated with this class of systems can be formulated as the problem of determining the parameters of the estimating system such that system performance is optimum, subject to the dynamic system constraints imposed by the plant and by the disturbance process.

Physical Considerations and Assumptions

It is assumed that the dynamic behavior of the plant can be adequately described by a system of linear differential equations. This assumption is made only in the interest of simplifying subsequent calculations and is not a restriction on the general synthesis technique.

It is assumed that only limited energy is available to the plant and that the energy consumed by the plant is a quadratic functional over the plant inputs $\underline{y}(t)$. A mathematical statement of this assumption will be incorporated into the system performance index to be discussed in Chapter IV.

The input and disturbance processes are in general random processes. It is assumed that the first two statistical moments for each random variable are known, and that the random variables can be adequately described in terms of the first two moments. Higher statistical moments can generally be neglected since in most practical engineering systems only average and mean square deviations of system performance from the ideal are of interest.

Finally, it is assumed that the disturbance process enters into the state of the plant in an additive fashion. That is, multiplicative disturbances are not considered.

Outline of the Synthesis Procedure

The synthesis procedure involves essentially the following steps.

(1) A mathematical model is constructed which adequately represents system dynamic behavior.

(2) An index of system performance is defined which provides a measure of quality of system design.

(3) The synthesis problem is posed as the variational problem of optimizing system performance with respect to

the performance index, subject to the constraints imposed by the dynamic system model.

(4) The variational problem is attacked by means of the dynamic programming method. Application of this method results in a functional equation which relates the parameters of the estimating system to the optimum value of the performance index.

(5) The functional equation is solved. The solution results in a specification of the configuration and parameters of the estimating system.

Notation

State variable notation* is used throughout this thesis. Using this notation, a dynamic system is described at any time by the value of its state vector. A state vector is denoted by a character underlined by a bar. Thus, $\underline{x}(t)$ denotes a state vector, and $x_i(t)$ denotes the i^{th} scalar component of the vector. In general, lower case symbols which are not underlined are used to denote scalar quantities. Capitalized symbols are used to represent matrices. Thus, $A(t)$ is a matrix with elements a_{ij}. The superscript T is used to denote the transpose of a vector or matrix. Thus, if

$$\underline{x}(t) = \begin{vmatrix} x_1 \\ \circ \\ \circ \\ \circ \\ x_n \end{vmatrix} \quad , \text{ then } \underline{x}^T = [x_1 \circ \circ \circ \circ x_n] \quad .$$

*See, for instance, Kalman (11).

The scalar or inner product of two vectors is referred to as a dot product. Thus, $\underline{x} \circ \underline{y} = \underline{x}^T \underline{y}$ is the dot product of the vector $\underline{x}$ with the vector $\underline{y}$ and $\underline{x} \circ \underline{y} = \underline{y} \circ \underline{x}$.

The symbol $E[\underline{x}]$ is used to denote the expected value of the vector $\underline{x}$ and expected value is defined as the ensemble average over all possible values of $\underline{x}$ at fixed time t. That is,

$$E[\underline{x}(t)] = \begin{vmatrix} Ex_1 \\ Ex_2 \\ \circ \\ \circ \\ Ex_n \end{vmatrix}$$

The covariance of the n vector $\underline{x}$ is defined by

$$E[\underline{x}(t_1), \underline{x}(t_2)] = E[\underline{x}(t_1)\underline{x}(t_2)^T] \quad ,$$

an nxn matrix.

The norm of a vector is denoted by norm $\underline{x} = \|x\|$ where

$$\|x\| = [x_1^2 + x_2^2 \circ\circ\circ + x_n^2]^{\frac{1}{2}} = [\underline{x} \circ \underline{x}]^{\frac{1}{2}} \quad .$$

A more detailed discussion of the state vector notation and its relationship to more conventional notation is given in Appendix I.

CHAPTER III

A MATHEMATICAL MODEL

In this chapter a general mathematical model is defined which is applicable to the class of systems represented by Figure 1. The mathematical model for the entire system defined here consists of a combination of subsystem mathematical models which represent the behavior of the plant and of the input and disturbance processes. The model for the entire system takes the form of a vector differential equation.*

The estimating system is initially unspecified and is described only in terms of its output vector $\underline{y}(t)$, an initially unspecified state vector. The plant is given as one of the design constraints so that its parameters are known. Thus, the plant output vector can be related to its input vector which is the same as the estimating system output vector. This relation constitutes a mathematical model for the internal system dynamics associated with the plant and estimating system. The input and disturbance processes are in general random and a mathematical model must be defined to represent these random processes. The combination of the models for the input and disturbance processes with that of the plant results in a mathematical model for the entire system.

*See Appendix I.

The Plant and Estimating System

The estimating system is initially unspecified, and its parameters are to be determined by the synthesis procedure. The dynamic behavior of the estimating system is described by its output state vector $\underline{y}(t)$ so that the estimating system is specified once $\underline{y}(t)$ is determined. Since the plant is specified, its parameters are known. By assumption, in the absence of external disturbance, the plant output vector can be related to the estimating system output vector by linear differential equations. In conventional notation, the plant outputs would be related to its inputs by a set of n^{th} order linear differential equations. More compact notation is possible using the concepts of state vectors and vector differential equations, as discussed in Appendix I. Thus, in the absence of external disturbances, the plant output vector $\underline{x}(t)$ is related to the estimating system output vector $\underline{y}(t)$ by the vector differential equation

$$\dot{\underline{x}}(t) = A^x(t)\underline{x}(t) + C_1(t)\underline{y}_1(t) \qquad (3.1)$$

where

$\underline{x}(t)$ is an (mxl) state vector
$\underline{y}_1(t)$ is an (mxl) input vector
$A^x(t)$ is an (mxm) matrix
$C_1(t)$ is an (mxm) matrix.

The superscript on the matrix $A^x(t)$ is used to denote the fact that the matrix is associated with the state variable $\underline{x}$.

All quantities in equation (3.1) are specified except the plant input $\underline{y}(t)$, which is to be synthesized. The m components of the plant state vector $\underline{x}(t)$ represent the instantaneous values of the quantities necessary to completely specify the plant dynamic behavior. In general, if an analog simulation of the plant is constructed, the m components of $\underline{x}(t)$ are the integrator outputs.

A Mathematical Model for a Random Process

As mentioned above, the input and disturbance processes are in general random. It is the purpose of this section to define an appropriate mathematical model for these random processes. The model used here is similar to that used by Bode and Shannon (3), but is not restricted to stationary random processes.

Subject to the previously mentioned assumption that only the first two statistical moments are required to desribe the random processes, a theorem by Doob (18) enables any random process to be represented by an appropriate Gaussian random process. Doob's theorem states that for any random process with given mean and covariance, there exists a Gaussian random process with identical mean and covariance. More specifically, if w(t) is a centered random variable, there exists a Gaussian random variable w(t) such that

$$E[\hat{w}(t)] = E[w(t)] = 0$$

$$E[\hat{w}(t)\hat{w}(s)] = E[w(t)w(s)] \quad .$$

Using this result, it can be shown that any random variable can be represented, insofar as its first two statistical moments are concerned, as a component of a vector random variable $\underline{w}_i(t)$ which satisfies

$$\dot{\underline{w}}_i(t) = A_i^w(t)\underline{w}_i(t) + B_i(t)\mathring{u}(t) \qquad (3.2)$$

where

$\underline{w}_i(t)$ is an $(q_i \times 1)$ state vector

A_i^w is a $(q_i \times q_i)$ matrix

B_i is a $(q_i \times 1)$ matrix

$\mathring{u}(t)$ is a scalar stationary white noise process.

In equation (3.2), the first component of $\underline{w}_i(t)$ is a realization of the Gaussian random variable $\hat{w}_i(t)$, and $\mathring{u}(t)$ is a stationary white noise process which will be discussed below. The elements of the matrices $A_i^w(t)$ and $B_i(t)$ are determined from the first two statistical moments of the given random process by methods outlined in Appendix III.

Equation (3.2) represents a description of a time varying filter excited by a stationary white noise process. Since the filter is time variable, the filter output $\underline{w}(t)$ is a non-stationary random process.

The white noise process $\mathring{u}(t)$ is formally the derivative of the elementary Gaussian process considered by Doob (19). Let u(t) denote Doob's elementary process, then u(t) has the properties

$$E[u(t) - u(s)] = 0$$

$$E[u(t) - u(s)]^2 = \sigma^2 |t-s| \quad .$$

Also, the u(t) process has independent increments. That is, if $t_1 < t_2 \cdots < t_n$, then the

$$\Delta u_i = [u(t_{i+1}) - u(t_i)] \quad (i = 1, 2, \cdots, n-1)$$

are mutually independent random variables. With probability one, u(t) is a continuous process. However, u(t) is not of bounded variation on any finite time interval so that $\frac{du}{dt}$ is undefined. However, for any continuous f(t), the Stieltjes sum

$$\int_{t_o}^{t} f(s)\ du\ (s)$$

exists and has the properties

$$E \int_{t_o}^{t} f(s)\ du\ (s) = 0$$

$$E[\int_{t_o}^{t} f(s) du(s)]^2 = \sigma^2 \int_{t_o}^{t} f^2(s)\ ds \quad . \tag{3.3}$$

Formally, du(t) can be written as $du(t) = \mathring{u}(t)\ dt$, and the Stieltjes sum as

$$\int_{t_o}^{t} f(s)\mathring{u}(s)ds \quad .$$

Then from (3.3) it follows that formally

$$E[\mathring{u}(t)\mathring{u}(s)] = \sigma^2\delta(t-s) \tag{3.4}$$

where $\delta(t)$ is the dirac delta function. Thus, $\mathring{u}(t)$ has the properties of a stationary white noise process with spectral density σ^2.

Although undefined rigorously, the $\mathring{u}(t)$ process will be used throughout this presentation as a formal device to avoid dealing with Stieltjes integrals. The white noise process is employed in a manner analogous to the use of impulse functions in deterministic systems.

As is shown in Appendix II, the Gaussian random variable $\hat{w}(t)$ can be represented as a linear transformation of the white noise process of the form

$$\hat{w}(t) = \int_{t_o}^{t} G(t,s)\mathring{u}(s)ds \tag{3.5}$$

where $G(t,s)$ is defined by equation (A-2.17). The random variable $\hat{w}(t)$ is then the output of a linear filter with impulse response $G(t,s)$ excited by the stationary white noise process $\mathring{u}(t)$. Since $G(t,s)$ is a time varying impulse response function, $\hat{w}(t)$ is a nonstationary random variable.

As is known*, the integral equation (3.5) is equivalent to an appropriate linear differential equation. For the

*Coddington and Levinson (20), pg. 193.

purposes of this work, it is expedient to convert the integral equation (3.5) to an equivalent differential equation representation in terms of the vector differential equation (3.2). In Appendix III, a procedure is outlined for performing the conversion from the integral representation to the differential equation. Using the methods of Appendix III, the matrix coefficients $A_i^w(t)$, and $B_i(t)$ of equation (3.2) are specified, thus defining the model for a random process represented by equation (3.2). Explicit specification of the matrices is given following equation (A-1.13) where the elements of A_i^w and B_i are determined from equations (A-1.15), and (A-3.16), respectively.

The Complete System Model

The complete system model consists of a combination of the subsystem models previously defined. Let $\underline{w}_1(t)$ be the plant disturbance process state vector, $\underline{w}_2(t)$ be the system input process state vector, and $\underline{x}(t)$ be the plant state vector. Then

$$\mathring{\underline{w}}_1(t) = A_1^w(t)\underline{w}_1(t) + B_1(t)\mathring{u}(t) \qquad (3.6)$$

$$\mathring{w}_2(t) = A_2^w(t)\underline{w}_2(t) + B_2(t)\mathring{u}(t) \qquad (3.7)$$

and since it is assumed that the disturbance process enters the plant state in an additive manner, the plant state in the presence of the disturbance process is specified by

$$\underline{\mathring{x}}(t) = A^{x}(t)\underline{x}(t) + D(t)\underline{w}_1(t) + C_1(t)\underline{y}_1(t) \tag{3.8}$$

where D(t) is an $(m \times q_1)$ matrix. Then by use of the definitions

$$\underline{z}(t) = \begin{vmatrix} \underline{x} \\ \underline{w}_1 \\ \underline{w}_2 \end{vmatrix} \qquad B(t) = \begin{vmatrix} 0 \\ B_1 \\ B_2 \end{vmatrix} \qquad \underline{y}(t) = \begin{vmatrix} y_1 \\ 0 \\ 0 \end{vmatrix}$$

$$A(t) = \begin{vmatrix} A^{x} & D & 0 \\ 0 & A_1^{w} & 0 \\ 0 & 0 & A_2^{w} \end{vmatrix} \qquad C(t) = \begin{vmatrix} C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} ,$$

equations (3.6), (3.7), (3.8) can be combined into the single vector differential equation

$$\underline{\mathring{z}}(t) = A(t)\underline{z}(t) + B(t)\mathring{u}(t) + C(t)\underline{y}(t) \quad . \tag{3.9}$$

In equation (3.9), $\underline{z}$ is an $(n=m+q_1+q_2 \times 1)$ state vector which represents the dynamic behavior of the complete system. Equation (3.9) then is a mathematical model for the system shown in Figure 1. In case all system parameters are deterministic, then $B(t) \equiv 0$ in equation (3.9).

CHAPTER IV

SYSTEM PERFORMANCE INDEX

In Chapter III, a mathematical model for the class of systems represented by Figure 1 was defined in terms of the vector differential equation (3.9). In equation (3.9), $\underline{y}(t)$ is an unknown vector function to be specified by the synthesis method. However, before a synthesis procedure can be formulated, it is necessary to define a system performance index which serves as a measure of the quality of the system. Various specific performance indices have been discussed in the literature, for instance the mean square error criterion introduced by Wiener (2). This chapter is devoted to defining a general class of performance indices applicable to the problem at hand. Necessary mathematical restrictions on the general class of indices are discussed. The synthesis problem is then posed as the variational problem of finding a $\underline{y}(t)$ which results in system performance which is optimum with respect to the performance index, subject to the system dynamic constraints imposed by equation (3.9).

The System Error Functional

An instantaneous measure of the deviation of system performance from the ideal can generally be defined as a scalar functional over the system state variables. That is, a system error functional $\varepsilon(t)$ can be defined by

$$\varepsilon(t) = F(\underline{z},\underline{y},t) \quad . \qquad (4.1)$$

As an example, if a quadratic error function is used in conjunction with a constraint on plant energy, then F is of the form

$$F(\underline{z},\underline{y},t) = \underline{z}\circ G\underline{z} + \underline{y}\circ Qy \qquad (4.2)$$

where G, Q are given symmetric matrices which weight the various components of the error functional. The first term in (4.2) represents the square of the deviation of the output state from the input and the second term is a measure of the energy supplied to the plant.

In order that the error functional have some physical significance, and to insure a solution to the synthesis problem, certain mathematical restrictions must be placed upon $F(\underline{z},\underline{y},t)$. In particular, if $\underline{z}_d$ represents the desired system state (the state which corresponds to zero system error), then it is required that

$$F(\underline{z}_d,\underline{y},t) = 0 \quad . \qquad (4.3)$$

Also for $\underline{z} \neq \underline{z}_d$, it is required that the error functional be a monotone increasing function of the deviation of $\underline{z}$ from $\underline{z}_d$ for both positive and negative deviations. That is, it is required that $F(\underline{z},\underline{y},t) \to \infty$ monotonically as $\|\underline{z} - z_d\| \to \infty$. In addition, it is required that the error functional be chosen such that first and second partial derivatives of $F(\underline{z},\underline{y},t)$ with respect to each component of $\underline{z}$ and $\underline{y}$ exist and such that the first partial derivative of $F(\underline{z},\underline{y},t)$ with respect to t is continuous on the measurement interval $[0,T]$.

It should be noted that choice of the error functional is dependent upon the use of the particular system under consideration. The error functional chosen for any given system is largely a matter of physical intuition on the part of the system designer. However, most physically meaningful measures of system error will conform to the general restrictions stated above.

The Performance Index

The performance index is a measure of system performance over the measurement interval. For this work, the performance index is taken to be a time dependent functional of the form

$$J(t) = \int_t^T F(\underline{z},\underline{y},s)ds \qquad 0 \leq t \leq T \quad . \tag{4.4}$$

That is, $J(t)$ is taken as a sum of the system error functional from current time t to the end of the measurement interval. Errors for time previous to current time t are not included in the performance index $J(t)$ since errors which have already occurred cannot be corrected and hence have no bearing on the current choice of plant inputs. Equation (4.4) describes the general class of performance indices to be considered. For specific systems, it is necessary to choose a specific form for the error functional such as that given by equation (4.2).

Optimum Performance

Since the performance index is defined as a measure of system error, it follows that best or optimum performance is obtained when J(t) assumes a minimum value. The choice of the plant input vector $\underline{y}(t)$ should then be made in such a manner as to minimize J(t). However, the fact that $\underline{z}(t)$ is in general a random vector precludes the possibility of finding a $\underline{y}(t)$ which minimizes J(t) for all possible values of $\underline{z}(t)$. A reasonable alternative is to attempt to find a $\underline{y}(t)$ such that the probability of J(t) exceeding a specified small value is small. That is, a $\underline{y}(t)$ will be sought which insures that

$$P[J(t) > a] < b \tag{4.5}$$

for some appropriate a,b. Using the Chebychev inequality*, the inequality (4.5) is satisfied if

$$E[J(t)] < N(a,b) \tag{4.6}$$

where N is some number depending upon a and b. The best hope for satisfying an inequality of the form (4.6) is to make the expected value of J(t) as small as possible. The synthesis problem then becomes one of determining a particular plant input vector $\underline{y}(t)$ such that the expected value of J(t) using $\underline{y}_{min}(t)$ is less than for any other possible plant input vector. Mathematically, the synthesis problem then becomes that of

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*Davenport and Root (21), pg. 63.

determining $\underline{y}_{min}(t)$ such that

$$E[J_{\underline{y}_{min}}(t)] \leq E[J_{y}(t)] \quad , \tag{4.7}$$

subject to the dynamic system constraints imposed by equation (3.9).

CHAPTER V

THE FUNCTIONAL EQUATION

In Chapter IV, the synthesis problem was posed as the variational problem of minimizing the expected value of the performance index expressed by equation (4.4), subject to the dynamic constraints imposed by equation (3.9). In this chapter, the minimization is attacked using the formalism of Bellman's dynamic programming method (12). The result of the application of this method is a functional equation which relates the minimum of the expected value of the performance index J(t) to the plant input vector, $\underline{y}(t)$. The limiting form of this functional equation is a partial differential equation.

Dynamic Programming

The variational problem posed in Chapter IV could be attacked by the classical Euler-Lagrange variational calculus. Application of this technique would require that the constraint equation (3.9) be converted to an equivalent integral equation so that the Lagrange multiplier method could be employed. Then application of the classical variational calculus would result in the Euler-Lagrange functional equation. The limiting form of this functional equation is a partial differential equation, the solution of which must satisfy two point boundary conditions. In general, the Euler-Lagrange equation is difficult to solve and requires special computational consideration.*

The method of dynamic programming, as developed by Bellman (12), provides an alternative approach to variational problems such as that posed in Chapter IV. Application of the dynamic programming method results in a functional equation, the solution of which is constrained to satisfy only one fixed boundary condition. Solution of this functional equation is conceptually much simpler than the solution of the functional equations which result from the Euler-Lagrange technique.

The formalism employed in the dynamic programming method follows directly from the application of the "principle of optimality." This principle** states that "an optimal policy has the property that, whatever the initial system state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the system state resulting from the first decision." In a simple geometric application, this principle states that any portion of an optimal trajectory is an optimal trajectory. A mathematical statement of this principle, as applied to the variational problem posed earlier, results in a functional equation relating the minimum of the expected value of the performance

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*Sagan (22), pg. 273.
**Bellman (12), pg. 83.

index to the plant input vector, $\underline{y}(t)$, as will be shown in the following.

Application of the Dynamic Programming Method

The statement of the principle of optimality involves an initial system state and an initial optimal decision. Thus to apply the optimality principle to the problem under consideration, consider an arbitrary time t_o which belongs to the measurement interval $[0,T]$. Let the system state at this time be denoted by $\underline{z}(t_o) = \underline{z}_o$. Assume that there exists a $\underline{y}(t_o)$ which gives $E[J(t_o)\underline{z}_o]$ a minimum value. For fixed T, the minimum value of the expected value of $J(t_o)$ given $\underline{z}_o$ is a function only of the time t_o and the system state $\underline{z}_o$ since the minimization over $\underline{y}$ eliminates the functional dependence upon the vector $\underline{y}$. Thus, if the minimum is denoted by the functional $\Psi(\underline{z}_o,t_o)$, then

$$\Psi(\underline{z}_o,t_o) = \min_{\underline{y}} E\{\int_{t_o}^{T} F(\underline{z},\underline{y},s)ds \mid \underline{z}_o\} \qquad (5.1)$$

The functional $\Psi(\underline{z}_o,t_o)$ is the minimum of the expected value of the performance index resulting from the initial decision, $\underline{y}(t_o)$, and can be used to develop a mathematical statement of the optimality principle.

The integral (5.1) can be expanded into the sum of two integrals as

$$\Psi(\underline{z}_o,t_o) = \min_{\underline{y}} E\{\int_{t_o}^{t_o+\Delta t} F(\underline{z},\underline{y},s)ds|\underline{z}_o + \int_{t_o+\Delta t}^{T} F(\underline{z},\underline{y},s)ds|\underline{z}_o\} \qquad (5.2)$$

where Δt is an arbitrary small time increment and the minimization operation is performed over the interval $[t_o, \tau]$. Referring to the differential equation (3.9), which governs the system state, it can be seen that whatever the value of $\underline{y}(t_o)$, its effect on system state on the interval $[t_o, t_o+\Delta t]$ is to change the state from $\underline{z}(t_o) = \underline{z}_o$ to $\underline{z}(t_o+\Delta t)$ or for Δt small enough to $\underline{z}(t_o+\Delta t) = \underline{z}_o+\Delta\underline{z}$. Then using the principle of optimality, the minimization of the second integral must be accomplished by a $\underline{y}(t)$ which is optimum with respect to the initial $\underline{y}(t_o)$ and $\underline{z}(t_o)$. The minimization of the second integral then becomes a problem identical to that which led to (5.1) so that (5.2) may be rewritten as

$$\Psi(\underline{z}_o, t_o) = \min_{\underline{y}} E\{\int_{t_o}^{t_o+\Delta t} F(\underline{z}, \underline{y}, s)ds + \Psi(\underline{z}_o+\Delta z, t_o+\Delta t)|\underline{z}_o\} \qquad (5.3)$$

where the minimization operation is performed over the interval $[t_o, t_o+\Delta t]$. Equation (5.3) is a mathematical statement of the optimality principle in this case.

Equation (5.3) provides a recurrence relationship between $\Psi(\underline{z}_o, t_o)$ and $\Psi(\underline{z}_o+\Delta\underline{z}, t_o+\Delta t)$. To make this relationship more useful, the term $\Psi(\underline{z}_o+\Delta\underline{z}, t_o+\Delta t)$ is expanded into a Taylor series as

$$\begin{aligned}\Psi(\underline{z}_o+\Delta\underline{z}, t_o+\Delta t) = {} & \Psi(\underline{z}_o, t_o) + \nabla_{\underline{z}_o} \Psi(\underline{z}_o, t_o)\circ\nabla\underline{z} + \\ & \frac{\partial\Psi(z_o, t_o)}{\partial t}\Delta t + \frac{\partial}{\partial t}(\nabla_{\underline{z}_o} \Psi(\underline{z}_o, t_o)\circ\Delta\underline{z}\Delta t + \\ & \frac{1}{2}\nabla^2_{\underline{z}_o} \Psi(\underline{z}_o, t_o)\Delta\underline{z}\circ\Delta\underline{z} + \frac{1}{2}\frac{\partial^2\Psi(z_o, t_o)}{\partial t^2}\Delta t^2 + R\end{aligned} \qquad (5.4)$$

where R is the remainder after the second variation and

$$\nabla_{\underline{z}} f(t) = \begin{vmatrix} \frac{\partial f}{\partial z_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f}{\partial z_n} \end{vmatrix} , \quad \nabla^2_{\underline{z}} f(t) = \begin{vmatrix} \frac{\partial^2 f}{\partial z_1^2} & \cdots & \frac{\partial^2 f}{\partial z_1 \partial z_n} \\ \vdots & \ddots & \\ \frac{\partial^2 f}{\partial z_n \partial z_1} & \cdots & \frac{\partial^2 f}{\partial z_n^2} \end{vmatrix} .$$

From equation (A-1.17), equation (3.9) has a solution of the form

$$\underline{z}(t) = \int_0^t \Phi(t)\Phi^{-1}(s)[B(s)\mathring{u}(s) + C(s)\underline{y}(s)]ds \quad . \qquad (5.5)$$

Thus the difference $\Delta\underline{z} = \underline{z}(t_o+\Delta t) - \underline{z}_o$ can be expressed as

$$\Delta\underline{z} = \int_{t_o}^{t_o+\Delta t} \Phi(t_o)\Phi^{-1}(s)[B(s)\mathring{u}(s) + C(s)\underline{y}(s)]ds \quad . \qquad (5.6)$$

Using equation (5.6), $\Delta\underline{z}$ can be replaced in equation (5.4) yielding

$$\begin{aligned} \Psi(\underline{z}_o,t_o) = \min_{\underline{y}}\{E(&\int_{t_o}^{t_o+t} F(\underline{z},\underline{y},s)ds + \Psi(\underline{z}_o,t_o) + \frac{\partial\Psi(\underline{z}_o,t_o)}{\partial t}\Delta t + \\ &[\nabla_{\underline{z}_o}\Psi(\underline{z}_o,t_o) + \Delta t\frac{\partial\nabla_{\underline{z}_o}\Psi(\underline{z}_o,t_o)}{\partial t_o}] \cdot \\ &[\int_{t_o}^{t_o+\Delta t}\Phi(t_o)\Phi^{-1}(s)[B(s)\mathring{u}(s) + C(s)\underline{y}(s)]ds] + \\ &[\tfrac{1}{2}\nabla^2_{\underline{z}_o}\Psi(\underline{z}_o,t_o)\int_{t_o}^{t_o+\Delta t}\Phi(t_o)\Phi^{-1}(s)[B(s)\mathring{u}(s) + C(s)\underline{y}(s)]ds] \cdot \\ &[\int_{t_o}^{t_o+\Delta t}\Phi(t_o)\Phi^{-1}(s)[B(s)\mathring{u}(s) + C(s)\underline{y}(s)]ds] + \\ &\frac{1}{2}\frac{\partial^2\Psi(\underline{z}_o,t_o)}{\partial t^2}\Delta t^2 + R)|\underline{z}_o\} \quad . \end{aligned} \qquad (5.7)$$

Then using the expected value operation in equation (5.7) results in

$$\Psi(\underline{z}_o, t_o) = \min_{\underline{y}}\{E[F(\underline{z}(t_1), t_1)] | \underline{z}_o + \Psi(\underline{z}_o, t_o) + \frac{\partial\Psi(\underline{z}_o, t_o)}{\partial t}\Delta t +$$
$$[\nabla_{\underline{z}_o}\Psi(\underline{z}_o, t_o) + \Delta t\frac{\partial\nabla_{t_o}\Psi(\underline{z}_o, t_o)}{\partial t_o}]\circ[\int_{t_o}^{t_o+\Delta t}\Phi(t_o)\Phi^{-1}(s)C(s)\underline{y}(s)ds] + \tag{5.8}$$
$$[\tfrac{1}{2}\nabla^2_{\underline{z}_o}\Psi(\underline{z}_o, t_o)\int_{t_o}^{t_o+\Delta t}\Phi(t_o)\Phi^{-1}(s)C(s)\underline{y}(s)ds]\cdot[\int_{t_o}^{t_o+\Delta t}\Phi(t_o)\Phi^{-1}(s)C(s)\underline{y}(s)ds] +$$
$$\tfrac{1}{2}\int_{t_o}^{t_o+\Delta t}B(s)\Phi^{-1}(s)\Phi(t_o)\nabla_{\underline{z}_o}\Psi(\underline{z}_o, t_o)\Phi(t_o)\Phi^{-1}(s)B(s)ds + \frac{1}{2}\frac{\partial^2\Psi(\underline{z}_o, t_o)}{\partial t^2}\Delta t^2 + R\}$$

In equation (5.8), the mean value theorem for integrals has been used on the integral $\int_{t_o}^{t_o+\Delta t}F(\underline{z}, \underline{y}, s)ds$ and t_1 is a time between t_o and $t_o+\Delta t$.

Now dividing equation (5.8) through by Δt, taking the limit as $\Delta t \to 0$, noting that

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{t_o}^{t_o+\Delta t}\Phi(t_o)\Phi^{-1}(s)C(s)\underline{y}(s)ds = A(t_o)\underline{z}_o + C(t_o)\underline{y}(t_o) \quad ,$$

and assuming that $\lim_{\Delta t\to 0}\frac{R}{\Delta t} = 0$ yields

$$0 = \min_{\underline{y}}[F(\underline{z}_o, \underline{y}_o, t_o) + \nabla_{\underline{z}_o}\Psi(\underline{z}_o, t_o)\circ[A(t_o)\underline{z}_o + C(t_o)y_o] +$$
$$\frac{\partial\Psi(\underline{z}_o, t_o)}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\Psi(z_o, t_o)}{\partial z_i\,\partial z_i}\,b_{m-j+1}\,b_{m-i+1}\,] \quad . \tag{5.9}$$

Equation (5.9) is a functional equation relating the minimum of the expected value of $J(t)$ to the plant input vector at time t_o. However, since t_o is an arbitrary time, equation

(5.9) holds for any time on the measurement interval [0,T]. Equation (5.9) then can be rewritten as

$$\frac{\partial \Psi(\underline{z},\tau)}{\partial \tau} = \min_{\underline{y}}\{F(\underline{z},\underline{y},\tau) + \nabla_z \Psi(\underline{z},\tau)\cdot[A(\tau)\underline{z}(\tau) + C(\tau)y(\tau)] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \Psi(z,\tau)}{\partial z_i \partial z_j} b_{m-i+1}\, b_{m-j+1}\} \tag{5.10}$$

where τ = T-t. The initial conditions for the partial differential equation (5.10) can be established from equation (5.1).

Equation (5.10) is the functional equation of the dynamic programming method. By performing the minimization with respect to $\underline{y}$, a relationship between the plant input vector $\underline{y}(t)$ and the functional $\Psi(\underline{z},\tau)$ is obtained. Then after eliminating $\underline{y}(t)$ from (5.10), a partial differential equation involving only $\Psi(\underline{z},\tau)$ results. This equation will be used in Chapter VII to obtain solutions for specific system error functionals.

It should be noted that for stationary systems when the interval [0,T] approaches infinity, the partial derivative $\frac{\partial \Psi}{\partial \tau}$ in equation (5.10) becomes zero. The computation considerations required to solve equation (5.10) are then much simpler. In a practical situation, this case would occur when the time constants associated with the system inputs are significantly larger than the longest plant time constant.

The fact that the functional $\Psi(\underline{z},\tau)$ which solves equation (5.10) provides a unique minimum of the performance index can be verified by a contradiction argument. Assume that there exists a different functional $g(\underline{z},\tau)$ which is smaller than $\Psi(\underline{z},\tau)$. That is, assume $g(\underline{z},\tau)$ satisfies

$$\frac{\partial g(z,\tau)}{\partial \tau} = \min_{\underline{y}}\{F(\underline{z},\underline{y},\tau) + \nabla_{\underline{z}} g(\underline{z},\tau)\cdot[A(\tau)\underline{z} + C\underline{y}] \quad (5.11)$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 g(z,\tau)}{\partial z_i \partial z_j} b_{m-j+1} b_{m-i+1}\}$$

with $g(\underline{z},0) = 0$, and $g(\underline{z},\tau) \leq \Psi(\underline{z},\tau)$.

Let $\underline{y} = \underline{y}^*$ be the vector which provides the minimum in equation (5.10), and $\underline{y} = \hat{\underline{y}}$ be the vector which provides the minimum in (5.11). Then from (5.10)

$$\frac{\partial \Psi}{\partial \tau} = F(\underline{z},\underline{y}^*,\tau) + \nabla_z \Psi\cdot[A\underline{z} + C\underline{y}^*] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 \Psi}{\partial z_i \partial z_j} b_{m-i+1} b_{m-j+1} \quad (5.12)$$

and from (5.11)

$$\frac{\partial g}{\partial \tau} = F(\underline{z},\hat{\underline{y}},\tau) + \nabla_z g\cdot[A\underline{z} + C\hat{\underline{y}}] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 g}{\partial z_i \partial z_j} b_{m-i+1} b_{m-j+1} \quad (5.13)$$

From equation (5.10), it follows that

$$\frac{\partial \Psi}{\partial \tau} \geq F(\underline{z},y,\tau) + \nabla_z g\cdot[A\underline{z} + C\hat{\underline{y}}] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 g}{\partial z_i \partial z_j} b_{m-i+1} b_{m-j+1} \quad (5.14)$$

but if (5.11) holds then

$$\frac{\partial g}{\partial \tau} \geq F(\underline{z},\underline{y}^*,\tau) + \nabla_z \Psi\cdot[A\underline{z} + C\underline{y}^*] + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 \Psi}{\partial z_i \partial z_j} b_{m-i+1} b_{m-j+1} \quad (5.15)$$

Using these inequalities, there results

$$\frac{\partial \Psi}{\partial \tau} - \frac{\partial g}{\partial \tau} \geq [\nabla_z \Psi - \nabla_z g]\cdot[A\underline{z} + C\hat{\underline{y}}] \geq [\nabla_z \Psi - \nabla_z g]\cdot[A\underline{z} + C\underline{y}^*] \quad . \tag{5.16}$$

To simplify (5.16), let f = -g then (5.16) becomes

$$\nabla_z f\cdot[A\underline{z} + C\hat{\underline{y}}] \geq \frac{\partial f}{\partial \tau} \geq \nabla_z f\cdot[A\underline{z} + C\underline{y}^*] \tag{5.17}$$

or as separate equations

$$\frac{\partial f}{\partial \tau} - \nabla_z f\cdot[A\underline{z} + C\hat{y}] \geq 0 \tag{5.18}$$

$$\frac{\partial f}{\partial \tau} - \nabla_z f\cdot[A\underline{z} + C\underline{y}^*] \leq 0 \quad .$$

But from (5.1), it follows that the minimum of the expected value of J(T) equals zero so that $f(\underline{z},0) = 0$. Then since the equations (5.18), with the equality holding, have identically zero solutions for $f(\underline{z},0) = 0$, it follows that

$$g = \Psi$$
$$y^* = \hat{y} \quad .$$

CHAPTER VI

SYSTEM STABILITY

The fact that the $\Psi(\underline{z},\tau)$, which furnishes a solution to the functional equation (5.10), minimizes the expected value of the performance index does not by itself guarantee that the resulting system is stable in the sense that the system output vector converges toward the input vector with time. It is, therefore, necessary to determine the conditions such that a stable system results from the synthesis procedure. As pointed out by Kalman (16), one method for doing this is to relate the functional $\Psi(\underline{z},\tau)$ to a suitable Lyapunov function. This section is devoted to establishing system stability criteria by means of Lyapunov stability theory. Since the systems under consideration are in general stochastic, only stability in the mean or average sense is investigated. Lyapunov stability theory has been interpreted for stochastic systems by Bertram and Sarachik (17) and the following discussion is based on their work.

Definition of Stability

System stability is a measure of the deviation of system motion from a specified or fixed trajectory. If $\underline{z}_d(t)$ denotes the desired system state (the state which results in zero for the expected value of system error), then the question of stability involves a measure of the difference between the expected value of the system state $\underline{z}(t)$ and $\underline{z}_d(t)$. More specifically, the following definitions of stability apply.

Definition 1. Stability

$\underline{z}_d$ is a stable in the mean system state if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for $\| z(t_o) \| < \delta(\varepsilon)$
$E[\| \underline{z}(t) - \underline{z}_d(t) \|] < \varepsilon \qquad t > t_o$.

Definition 2. Asymptotic Stability

$\underline{z}_d$ is asymptotically stable in the mean if it is stable (Definition 1) and for each $t_o \varepsilon [0, \infty]$ there exists a $\delta(t_o) > 0$ such that $\lim_{t \to \infty} E[\| \underline{z}(t) - \underline{z}_d(t) \|] \to 0$ whenever $E[\| \underline{z}(t_o - \underline{z}_d(t_o) \|] < \delta(t_o) \qquad t > t_o$.

For most engineering application, the stronger asymptotic stability is required since stability in the sense of definition one allows oscillation about the specified state.

The application of the Lyapunov stability theory to a given system consists of defining a Lyapunov function with properties which imply the desired type of stability. Asymptotic stability for the class of systems under consideration is implied by the existence of a Lyapunov function which satisfies the following theorem.

Theorem. (Bertram and Sarachik (17))

If there exists a Lyapunov function $V(\underline{z}, t)$ such that

a. $V(\underline{z}_d, t) = 0$

b. $V(\underline{z},t)$ is continuous in $\underline{z}$ and t and the first partial derivatives of $V(\underline{z},t)$ with respect to each component of $\underline{z}$ and t exists.

c. $V(\underline{z},t) \geq \alpha(\|\underline{z}\|) > 0 \qquad \underline{z} \neq \underline{z}_d$.

d. There exists a continuous, nondecreasing function $\beta(\|\underline{z}\|)$ such that $V(\underline{z},t) \leq \beta(\|\underline{z}\|)$.

e. $E\{\frac{dV(z,t)}{dt}\} \leq -g(\|\underline{z}\|)$ where $g(\|\underline{z}\|)$ has the properties $g(\|\underline{z}_d\|) = 0$, $g>0$ otherwise, and $g(\|\underline{z}\|)\to\infty$ monotonically at $\|\underline{z}-\underline{z}_d\|\to\infty$. Then, $\underline{z}_d$ is an asymptotically stable in the mean system state.

The conditions necessary for a stable system can now be determined in terms of the above stated theorem. The general procedure involved in determining these necessary conditions is to relate the functional $\Psi(\underline{z},t)$ discussed in Chapter V to an appropriate Lyapunov function.

To relate the functional $\Psi(\underline{z},t)$ to the Lyapunov function $V(\underline{z},t)$, consider the differential equation

$$\min_{\underline{y}} E\{\frac{dV(z,t)}{dt} + F(\underline{z},\underline{y},t)\,|\,z_o\} = 0 \quad . \tag{6.1}$$

Since the error functional $F(\underline{z},\underline{y},t)$ was restricted in Chapter IV to be greater than zero for $\underline{z} \neq \underline{z}_d$ and equal to zero for $\underline{z} = \underline{z}_d$, it follows that the $V(\underline{z},t)$ which solves equation (6.1) satisfies condition (e) of the theorem. Upon integrating equation (6.1) from t_o to ∞ there results

$$V(\underline{z}_o, t_o) = \min_{\underline{y}} E\{\int_{t_o}^{\infty} F(\underline{z}, \underline{y}, s)ds | \underline{z}_o\} \quad . \tag{6.2}$$

From the restrictions imposed upon the functional $F(\underline{z},\underline{y},t)$ in Chapter IV, the $V(\underline{z}_o,t_o)$ in equation (6.2) satisfies conditions (a) and (b) of the theorem. Also, since $F(\underline{z},\underline{y},t)$ is always greater than or equal to zero, condition (c) is also satisfied by $V(\underline{z}_o,t_o)$. Since t_o is arbitrary, these conditions are also satisfied for any t in the interval $[0,\infty]$. It follows from the theorem then that if $V(\underline{z},t)$ also satisfies condition (d) of the theorem, then $V(\underline{z},t)$ is an appropriate Lyapunov function.

Now by comparing equation (6.2) with equation (5.1), it follows that

$$V(\underline{z},t) = \Psi(\underline{z},t)\Big|_{T=\infty} \quad . \tag{6.3}$$

Consequently, if the $(\underline{z},t)$ which furnishes a solution to the functional equation (5.10) satisfies condition (d) of the theorem for $T = \infty$, then the system resulting from this solution is asymptotically stable in the mean.

CHAPTER VII

SYSTEM SYNTHESIS

In Chapter IV, the error functional $F(\underline{z},\underline{y},t)$ has been discussed in general terms. To design a system for a particular application, however, it is necessary to select a specific error functional and corresponding performance index. In this chapter, the functional equation (5.10) is solved for two important classes of performance indices which satisfy the conditions specified in Chapter IV. These are a minimum squared error performance index, and a final value performance index. In both cases it is assumed that a constraint on the plant input energy exists.

Minimum Squared Error

Consider a system described by equation (3.9). Let $\underline{w}_2$ be the system input vector, and $\underline{x}$ the plant output vector. The instantaneous value of the square of the difference between the respective components of $\underline{w}_2$ and $\underline{x}$ can be expressed as

$$\varepsilon^2 = (\underline{x}-w_2)^T \cdot G'(\underline{x}-\underline{w}_2) \tag{7.1}$$

where G' is a continuous symmetric matrix which weights the various elements of ε^2. For instance, if only the error between the first components of $\underline{w}_2$ and $\underline{x}$ is of interest then

G' is a matrix with

$$g'_{ij} = \begin{cases} 1 & i=j=1 \\ 0 & \text{otherwise} \end{cases} \quad .$$

The difference $(\underline{x} - \underline{w}_2)$ can be expressed in terms of the system state vector $\underline{z}$ as

$$\underline{x} - \underline{w}_2 = \underline{g}^T \underline{z} \qquad (7.2)$$

where $\underline{g}$ is a continuous n vector. For instance, with $\underline{z}$ defined as in equation (3.9), $\underline{g}$ is a vector with elements one through m and m + g_1 + 1 through n equal to one and zero otherwise. Using equation (7.2), the difference squared expressed by equation (7.1) can be written in terms of the system state vector, $\underline{z}(t)$ as

$$\varepsilon^2 = \underline{z} \cdot G\underline{z} \qquad (7.3)$$

where

$$G = \underline{g}\, G' \underline{g}^T \quad . \qquad (7.4)$$

The constraint on plant input energy can be taken into account by considering that expenditure of plant input energy contributes to the system error. The system error functional is then chosen as

$$F(\underline{z},\underline{y},t) = \underline{z} \cdot G\underline{z} + \lambda^2\, \underline{y} \cdot Q\underline{y} \qquad (7.5)$$

where Q is a continuous symmetric matrix.

The first term in equation (7.5) is the square of the difference between the various components of the input and output vectors and the second term is a measure of the plant input energy. The coefficient λ is an arbitrary design parameter which weights the relative importance of the two terms.

In accordance with equation (4.4), the system performance index for the minimum squared error system is given by

$$J(t) = \int_t^T (\underline{z}\cdot G\underline{z} + \lambda^2 \underline{y}\cdot Q\underline{y})dt \quad . \tag{7.6}$$

To determine the particular $\underline{y}(t)$ which minimizes the expected value of equation (7.6), the error functional defined by equation (7.5) is substituted into equation (5.10) to obtain

$$\frac{\partial \Psi}{\partial \tau} = \min_{\underline{y}} \{\underline{z}\cdot G\underline{z} + \lambda^2 \underline{y}\cdot Q\underline{y} + \nabla_{\underline{z}}\Psi\cdot[A\underline{z} + C\underline{y}]$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \Psi}{\partial z_i \partial z_j} b_{m-i+1} b_{m-j+1} \tag{7.7}$$

From equation (7.6), the initial conditions for this partial differential equation are established as $\Psi(\underline{z}, =0) = 0$. Solution of this functional equation will result in a specification of the parameters of a system which is optimum with respect to the minimum error squared performance index.

Considering first the minimization operation over $\underline{y}$, a necessary condition for a minimum to exist is that the partial derivative of equation (7.7) with respect to each component of $\underline{y}$ be zero. In vector notation, this necessary condition can be expressed by the requirement that

$$\nabla_{\underline{y}} \frac{\partial \Psi}{\partial \tau} = 0. \tag{7.8}$$

Using equation (7.8) in equation (7.7) results in the condition that the particular $\underline{y}(t)$ which yields the minimum in equation (7.7) is

$$\underline{y}_{min} = -\frac{1}{2\lambda^2} Q^{-1} C^T \nabla_{\underline{z}} \Psi \quad . \tag{7.9}$$

That is, the $\underline{y}$ which satisfies equation (7.9) yields a minimum value for equation (7.7). As shown in Chapter V, the minimum provided by this particular $\underline{y}$ is unique.

Using equation (7.9), $\underline{y}$ can be eliminated from equation (7.7) to yield the partial differential equation in $\Psi(\underline{z},\tau)$

$$\frac{\partial \Psi}{\partial \tau} = \underline{z}\, G\underline{z} - \frac{1}{4\lambda^2} CQ[\overset{-1}{(}C\nabla_{\underline{z}}\Psi)\cdot\nabla_{\underline{z}}\Psi] + \nabla_{\underline{z}}\Psi\cdot A_{\underline{z}}$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\Psi}{\partial z_i \partial z_j} b_{m-i+1}\, b_{m-j+1} \tag{7.10}$$

with initial condition

$$\Psi(\underline{z},0) = 0.$$

The equation (7.9) determines the plant input vector $\underline{y}(t)$ as an explicit function of Ψ. Thus, to completely specify the vector $\underline{y}(t)$, it is necessary to solve equation (7.10) explicitly for $\Psi(\underline{z},\tau)$. To obtain a solution for equation (7.10), an approach similar to Bernoulli's separation method* is used. That is, the functional $\Psi(\underline{z},\tau)$ is expanded into a power series in such a manner that the partial differential equation (7.10) becomes a set of ordinary differential equations. That is, $\Psi(\underline{z},\tau)$ is expanded as

$$\Psi(\underline{z},\tau) = \Psi_o(\tau) + \Psi_1(\tau)\cdot\underline{z} + \Psi_2\underline{z}\cdot\underline{z} + \cdots \qquad (7.11)$$

where Ψ_o is a scalar, Ψ_1 an n vector, Ψ_2 an nxn matrix, etc.. Then

$$\frac{\partial\Psi}{\partial\tau} = \frac{\partial\Psi_0}{\partial\tau} + \frac{\partial\Psi_1}{\partial\tau}\underline{z} + \frac{\partial\Psi_2}{\partial\tau}\underline{z}\cdot\underline{z} + \cdots \quad , \qquad (7.12)$$

$$\nabla_{\underline{z}}\Psi = \Psi_1 + (\Psi_2 + \Psi_2^T)\underline{z} + \cdots$$

$$\frac{\partial^2\Psi}{\partial z_i \partial z_j} = \text{trace } \Psi_2 + \cdots \quad .$$

Upon substituting equation (7.11) into (7.10), it is found that only the first three terms in the series (7.11) are required since all higher terms are zero, and that the matrix Ψ_2 may be symmetric so that $\Psi_2 = \Psi_2^T$. The equation that results from this substitution is then

$$\left(\frac{\partial\Psi_0}{\partial\tau} + \frac{\partial\Psi_1}{\partial\tau}\cdot\underline{z} + \frac{\partial\Psi_2}{\partial\tau}\underline{z}\cdot\underline{z}\right) = (G^T\underline{z})\cdot\underline{z} - \frac{1}{4\lambda^2}\,[(CQ^{-1}C^T\Psi_1)\cdot\Psi_1 +$$

$$(4\Psi_2CQ^{-1}C^T\Psi_1)\cdot\underline{z} + (4\Psi_2CQ^{-1}C^T\Psi_2\underline{z})\cdot\underline{z}] +$$

$$A^T\Psi_1\cdot\underline{z} + (2A^T\Psi_2\underline{z})\cdot\underline{z} + \sum_{i=1}^{n}\frac{1}{2}\Psi_{2_{ii}}\, b^2_{m-i+1} \quad . \qquad (7.13)$$

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*op. cit. (22), pg. 91.

Equating like powers of $\underline{z}$ in (7.13) results in the set of ordinary differential equations

$$\dot{\Psi}_o(\tau) = -\left(\frac{1}{4\lambda^2} CQ^{-1}C^T\Psi_1\right)\cdot\Psi_1 + \sum_{i=1}^{n} \frac{1}{2}\Psi_{2_{ii}} b_{m-i+1} \tag{7.14}$$

$$\dot{\Psi}_1(\tau) = -\frac{1}{\lambda^2}\Psi_2 CQ^{-1}C^T\Psi_1 + A^T\Psi_1 \tag{7.15}$$

$$\Psi_2\underline{z}\cdot\underline{z} = (G^T\underline{z})\cdot\underline{z} - \left(\frac{1}{\lambda^2}\Psi_2 CQ^{-1}C^T\Psi_2\underline{z}\right)\cdot\underline{z} + (2A^T\Psi_2\underline{z})\cdot\underline{z} \tag{7.16}$$

where $\dot{\Psi}_i$ denotes differentiation with respect to τ. Since $\Psi(\underline{z},0) = 0$, the initial conditions for this set of equations are $\Psi_o(0) = \Psi_1(0) = \Psi_2(0) = 0$. With $\Psi_1(0) = 0$, the differential equation (7.15) has only the trivial solution $\Psi_1(\tau) = 0$. Thus the functional $\Psi(z,\tau)$ is completely determined by the solutions of the differential equations (7.14) and (7.16) with $\Psi_1(\tau) = 0$. The equation (7.9) can then be written explicitly as

$$\underline{y}_{min} = -\frac{1}{\lambda^2} Q^{-1}C^T\Psi_2\underline{z} \quad . \tag{7.17}$$

Using equation (7.17) in conjunction with equation (7.16), the structure and parameters of the estimating system can be completely specified. Thus, the synthesis procedure is completed. Equation (7.17) specifies that plant input or estimating system output $\underline{y}(t)$ as a function of the matrix $\Psi_2(\tau)$ and the system state variable $\underline{z}$. The solution to the differential equations (7.16) determines $\Psi_2(\tau)$ so that $\underline{y}(t)$ is explicitly determined as a time varying gain matrix multiplying the system state vector $\underline{z}$. Thus the structure of the synthesized system is a linear feedback system with time varying gains.

Since the differential equations (7.16) are nonlinear in general, either analog or digital computer is required to implement the solution to these equations. Standard computing techniques are available for determining numerical solutions to such equations.

To accommodate the time variable coefficients determined by the elements of the matrix Ψ_2, it is necessary to use time variable elements in the realization of the system specified by equation (7.17). Many such elements can be realized using standard computing equipment

As pointed out in Chapter V, in the case of stationary systems with infinite measurement intervals, the derivative $\frac{\partial \Psi}{\partial \tau}$ in equation (5.10) is zero. In this case then, the equations (7.16) reduce to a set of nonlinear algebraic equations so that the computational procedure is simplified. The synthesized system in this case can be realized with time invariant elements.

Final Value Systems

In many cases, systems are designed to produce a measurement of the input vector at only a particular time T. Such systems are referred to as final value systems. Taking the plant input energy constraint into account, a performance index for such systems can be taken as

$$J(t) = \underline{z}(T)\cdot G(T)\underline{z}(T) + \int_t^T \lambda^2 \underline{y}\cdot Q\underline{y}dt \quad . \tag{7.18}$$

This performance index is very similar to the minimum squared error index. The exception being that the term relating the errors between the various components of the input and output vectors depends only upon the final time T.

Using equation (7.18), the functional equation (5.10) in this case becomes

$$\frac{\partial\Psi}{\partial\tau} = \min_{\underline{y}}[\lambda^2\underline{y}\cdot Q\underline{y} + \nabla_{\underline{z}}\Psi\cdot(A_{\underline{z}} + C\underline{y}) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\Psi}{\partial z_i\partial z_j}b_{m-i+1}b_{m-j+1}] \ . \tag{7.19}$$

The initial conditions for equation (7.19) are established from equation (7.18) as

$$\Psi(\underline{z},0) = \underline{z}(T)\cdot G(T)\underline{z}(T) \quad .$$

Using equation (7.8), the particular $\underline{y}$ which provides the minimum in equation (7.19) is again

$$\underline{y}_{min} = -\frac{1}{2\lambda^2}Q^{-1}C^T\nabla_{\underline{z}}\Psi \quad . \tag{7.20}$$

Then using equation (7.20) in equation (7.19) results in

$$\frac{\partial\Psi}{\partial\tau} = -\frac{1}{4\lambda^2}CQ^{-1}[(C\nabla_{\underline{z}}\Psi)\cdot\nabla_{\underline{z}}\Psi] + \nabla_{\underline{z}}\Psi\cdot A_{\underline{z}} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\Psi}{\partial z_i\partial z_j}b_{m-i+1}b_{m-j+1} \ . \tag{7.21}$$

Again using the power series (7.11) for $\Psi(\underline{z},\tau)$ results in the set of ordinary differential equations

$$\dot{\Psi}_o(\tau) = -[\frac{1}{4\lambda^2}CQ^{-1}C^T\Psi_1]\cdot\Psi_1 + \sum_{i=1}^{n}\frac{1}{2}\Psi_{2_{ii}}b_{m-i+1} \tag{7.22}$$

$$\dot{\Psi}_1(\tau) = -\frac{1}{\lambda^2}\Psi_2 C Q^{-1} C^T \Psi_1 + A^T \Psi_1 \qquad (7.23)$$

$$\dot{\Psi}_2 \underline{z} \cdot \underline{z} = -[\frac{1}{\lambda^2}\Psi_2 C Q^{-1} C^T \Psi_2 \underline{z}] \cdot \underline{z} + [2A^T \Psi_2 \underline{z}] \cdot \underline{z} \qquad (7.24)$$

The initial conditions for this set of equations are

$$\Psi(0) = \Psi_1 = 0, \ \Psi_2(0) = G^T(T) \quad .$$

Again, only the trivial solution $\Psi_1 = 0$ results from equation (7.23) with $\Psi_1(0) = 0$, so that $\Psi(\underline{z},\tau)$ is determined by equations (7.22) and (7.24). The plant input vector is then determined as an explicit function of $\underline{z}$ by

$$\underline{y}_{min} = -\frac{1}{\lambda^2} Q^{-1} C^T \Psi_2 \underline{z} \quad . \qquad (7.25)$$

Equation (7.5) is of the same form as equation (7.17) which resulted from the minimum squared error performance index. However, the matrix Ψ_2 is different in each case.

Equation (7.25) in conjunction with equation (7.22) and (7.24) completely determine the system parameters of the final value system. The system structure for this case is very similar to that resulting from the minimum error squared performance index.

Stability

The general stability requirements for the class of systems under consideration have been discussed in Chapter VI. It was shown that a stable system results from the synthesis procedure if the functional $\Psi(\underline{z},\tau)$ is bounded from above for $T = \infty$. In the particular cases considered in this chapter, this condition can be made more specific.

The functional $\Psi(\underline{z},\tau)$ which solves either equation (7.7) or (7.19) can be written as

$$\Psi(\underline{z},\tau) = \Psi_o(\tau) + \Psi_2(\tau)\underline{z}\cdot\underline{z} \geq 0 \quad . \tag{7.26}$$

Also, due to the continuity restrictions placed upon the matrices G and Q, the performance index J(t) defined by either equation (7.6) or (7.18) is continuous. Consequently, the corresponding $\Psi(\underline{z},\tau)$ in either case is also continuous for any τ in the interval $[0,\infty]$. Thus, the quadratic form (7.26) can be bounded by an appropriate $\beta(\|\underline{z}\|)$. For instance,

$$\beta(\|\underline{z}\|) = \beta_1(\tau) + \beta_2(\tau)\underline{z}\cdot\underline{z} = \beta_1(\tau) + \beta_2(\tau)\|\underline{z}\|^2$$

with $\beta_1 \geq \Psi_o$, $\beta_2 \geq \Psi_2$ is an upper bound on $\Psi(\underline{z},\tau)$ for any τ on $[0,\infty]$. Consequently, the stability of the system resulting from the minimization of the expected values of the performance indices considered in this chapter is insured.

CHAPTER VIII

EXAMPLES

The purpose of this chapter is to present examples which illustrate the procedures outlined in the preceding chapters. Examples one and two illustrate the construction of mathematical models for random processes as outlined in Chapter III. Examples three through five illustrate the procedures involved in the synthesis method for the performance indices discussed in Chapter VII. These examples are simplified for the sake of brevity, but illustrate the general procedure required for more complex problems.

The numerical results presented in this chapter were obtained using a digital computer.

Example One. A Mathematical Model For a Stationary Random Process

Consider a stationary random variable w with zero mean and covariance

$$r_{ww}(t_1,t_2) = 4e^{-(t'-t)} + 2e^{-3(t'-t)} \qquad (8.1)$$

where $t' = \max(t_1,t_2)$, $t = \min(t_1,t_2)$. The following is devoted to the construction of a mathematical model in the form of the vector differential equation (3.2) for this random process.

The equation (8.1) is of the form of the sum (A-3.5). As shown in Appendix III, the functions $\phi_1(t) = 2e^{-t}$, $\phi_2(t) = \sqrt{2e}^{-3t}$ constitute a fundamental set of solutions for the homogeneous differential equation $L_t w = 0$. Then, from (A-1.4)

$$L_t w = \det \begin{vmatrix} w & 2e^{-t} & \sqrt{2e}^{-3t} \\ \dot{w} & -2e^{-t} & -3\sqrt{2e}^{-3t} \\ \ddot{w} & 2e^{-t} & 9\sqrt{2e}^{-3t} \end{vmatrix} \quad . \tag{8.2}$$

Expanding the determinant (8.2) results in the homogeneous differential equation

$$L_t w = \ddot{w} + 4\dot{w} + 3w = 0 \quad . \tag{8.3}$$

Thus, the coefficients of the operator L_t are $p_o = 3$, $p_1 = 4$. In the alternate vector representation then the elements of the matrix A^w are found from equation (A-1.15) as $a_1 = p_1 = 4$, $a_o = p_o = 3$. From (A-1.16) the fundamental matrix for the homogeneous vector differential equation $\dot{\underline{w}} = A^w(t)\underline{w}$ is then

$$\Phi(t) = \begin{vmatrix} 2e^{-t} & \sqrt{2e}^{-3t} \\ 6e^{-t} & \sqrt{2e}^{-3t} \end{vmatrix} \quad . \tag{8.4}$$

As in equation (A-3.9), the covariance matrix for the vector random variable $\underline{w}(t)$ is

$$R_{\underline{ww}}(t_1,t_2) = \Phi(t')D(t)\Phi^T(t) \tag{8.5}$$

From equation (A-3.11) the elements of the matrix D(t) are

$$d_{12} = d_{21} = 0$$

$$d_{11} = e^{2t}$$

$$d_{22} = e^{6t}$$

so that

$$R_{\underline{ww}}(t_1, t_2) = \begin{vmatrix} 2e^{-t'} & \sqrt{2e}^{-3t'} \\ 6e^{-t'} & \sqrt{2e}^{-3t'} \end{vmatrix} \begin{vmatrix} 2e^{t} & 6e^{t} \\ \sqrt{2e}^{3t} & \sqrt{2e}^{3t} \end{vmatrix} . \qquad (8.6)$$

Now, using equation (A-3.13), the difference function $\Delta(t_1, t_2)$ is given by

$$\Delta(t_1, t_2) = \begin{vmatrix} 2e^{-t_1} & \sqrt{2e}^{-3t_1} \\ 6e^{-t_1} & \sqrt{2e}^{-3t_1} \end{vmatrix} \begin{vmatrix} 2e^{t_2} & 6e^{t_2} \\ \sqrt{2e}^{3t_2} & \sqrt{2e}^{3t_2} \end{vmatrix} -$$

$$\begin{vmatrix} 2e^{t_1} & \sqrt{2e}^{3t_1} \\ 6e^{t_1} & \sqrt{2e}^{3t_1} \end{vmatrix} \begin{vmatrix} 2e^{-t_2} & 6e^{-t_2} \\ \sqrt{2e}^{-3t_2} & \sqrt{2e}^{-3t_2} \end{vmatrix} . \qquad (8.7)$$

Then from (A-3.15), the elements of the matrix B(t) in equation (3.9) are determined by

$$-b_1^2(t) = \frac{\partial}{\partial t_1}[4e^{-(t_1-t_2)} + 2e^{-3(t_1-t_2)} - 4e^{+(t_1-t_2)} - 2e^{3(t_1-t_2)}]\Big|_{t_1 = t_2} = -20$$

and

$$-b_1^2(t) = \frac{\partial}{\partial t_1}[36e^{-(t_1-t_2)} + 2e^{-3(t_1-t_2)} - 36e^{(t_1-t_2)} - 2e^{+3(t_1-t_2)}]\Big|_{t_1 = t_2} = -84 .$$

A mathematical model for the given random process in the form equation (3.9) applies with

$$A^{w}(t) = \begin{vmatrix} -4 & 1 \\ -3 & 0 \end{vmatrix} , \quad B(t) = \begin{vmatrix} \sqrt{20} \\ \sqrt{84} \end{vmatrix} .$$

This result can be readily verified using conventional spectral factorization techniques.

Example Two. A Mathematical Model For a Nonstationary Random Process

Consider the nonstationary random process with realization $w(t)$ which has zero mean and covariance

$$r_{ww}(t_1, t_2) = \frac{1}{t'} + \frac{t}{4t'^2} \quad . \tag{8.8}$$

The fundamental set for the homogeneous differential $L_t w = 0$ are

$$\phi_1(t) = \frac{1}{t}$$

$$\phi_2(t) = \frac{1}{2t^2} \quad .$$

The operator L_t is then specified by

$$L_t w = \det \begin{vmatrix} w & 1/t & 1/2t^2 \\ \dot{w} & -1/t^2 & -1/t^3 \\ \ddot{w} & 2/t^3 & 3/t^4 \end{vmatrix} = 0 \quad . \tag{8.9}$$

Expanding the determinant (8.9) results in

$$L_t(w) = \ddot{w} + \frac{4}{t}\dot{w} + \frac{2}{t^2}w = 0 \quad . \tag{8.10}$$

The coefficients of the operator L_t are then

$$p_1 = \frac{4}{t}$$

$$p_0 = \frac{2}{t^2} \quad .$$

Then from equation (A-1.15) the elements of the matrix $A^w(t)$ are

$$a_1(t) = p_1(t) = \frac{4}{t}$$

$$a_o(t) = p_o(t) - \dot{a}_1(t) = \frac{2}{t^2} - \frac{4}{t^2} = \frac{2}{t^2} \quad .$$

The matrix $A^w(t)$ is then given by

$$A^w(t) = \begin{vmatrix} -4/t & 1 \\ 2/t^2 & 0 \end{vmatrix} \quad . \tag{8.11}$$

The fundamental matrix for the homogeneous vector differential equation $\dot{\underline{w}} = A^w(t)\underline{w}$ is

$$\Phi(t) = \begin{vmatrix} 1/t & 1/2t^2 \\ 3/t^2 & 1/t^3 \end{vmatrix} \quad . \tag{8.12}$$

The covariance matrix for the random vector $\underline{w}(t)$ is

$$R_{\underline{ww}}(t_1,t_2) = \Phi(t')D(t)\Phi^T(t)$$

and, using equation (A-3.11), the elements of the matrix D(t) are

$$d_{12} = d_{21} = 0$$

$$d_{11} = \frac{1}{1/t} = t$$

$$d_{22} = \frac{t/2}{1/2t^2} = t^3 \quad .$$

Then

$$R_{\underline{ww}}(t_1,t_2) = \begin{vmatrix} 1/t' & 1/2t^{12} \\ 3/t^{12} & 1/t^{13} \end{vmatrix} \begin{vmatrix} 1 & 3/t \\ t/2 & 1 \end{vmatrix} . \qquad (8.13)$$

From equation (A-3.13) the difference $\Delta(t_1,t_2)$ is

$$\Delta(t_1,t_2) = \begin{vmatrix} 1/t_1 & 1/2t_1^2 \\ 3/t_1^2 & 1/t_1^3 \end{vmatrix} \begin{vmatrix} 1 & 3/t^2 \\ t^2/2 & 1 \end{vmatrix} -$$

$$\begin{vmatrix} 1 & t'/2 \\ 3t_1 & 1 \end{vmatrix} \begin{vmatrix} 1/t_2 & 3/t_2^2 \\ 1/2t_2^2 & 1/t_2^3 \end{vmatrix} . \qquad (8.14)$$

Then using (A-3.15)

$$-b_1^2(t_1) = \frac{\partial}{\partial t_1}\left[\frac{1}{t_1} + \frac{t_2}{4t_1^2} - \frac{1}{t_2} - \frac{t_1}{4t_2^2}\right]\Big|_{t_1=t_2} = -\frac{7}{4t^2}$$

and

$$-b_o^2(t_1) = \frac{\partial}{\partial t_1}\left[\frac{9}{t_1^2 t_2} + \frac{1}{t_1^3} - \frac{9}{t_1 t_1^2} - \frac{1}{t_2^3}\right]\Big|_{t_1=t_2} = -\frac{12}{t^4} .$$

A model in the form of equation (3.9) then applies with $A^w(t)$ given by equation (8.11) and $B(t)$ by

$$B(t) \quad \begin{vmatrix} \sqrt{7}/2t \\ \sqrt{12}/t^2 \end{vmatrix} .$$

<u>Example Three. A Deterministic System</u>

The design of a simple deterministic system is considered here. Three different cases are considered and simulated system performance is presented graphically. The specified system plant used throughout is governed by the scalar differential equation

$$\alpha\ddot{x}_o + \dot{x}_o = y_i \tag{8.15}$$

where x_o is the plant output and y_i is the plant input to be determined.

Case One. A Constant Input with Squared Error Performance Index

Without loss of generality, the constant system input is taken as $x_i = 1$. Then making the identification

$$z_1 = x_o - x_i = x_o - 1$$

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = \frac{-z_2}{\alpha} + \frac{y_i}{\alpha}$$

the mathematical model for the system can be expressed as

$$\dot{\underline{z}} = A\underline{z} + C\underline{y} \tag{8.16}$$

where

$$\underline{z} = \begin{vmatrix} z_1 \\ z_2 \end{vmatrix}, \quad \underline{y} = \begin{vmatrix} 0 \\ y_i \end{vmatrix}, \quad A = \begin{vmatrix} 0 & 1 \\ 0 & -1/\alpha \end{vmatrix}, \quad C = \begin{vmatrix} 0 & 0 \\ 0 & 1/\alpha \end{vmatrix}.$$

The performance index is taken as

$$J(t) = \int_t^T [(x_o - x_i)^2 + \lambda^2 y_i^2]dt \tag{8.17}$$

which can be expressed in vector notation as

$$J(t) = \int_t^T \underline{z}\cdot G\underline{z} + \lambda^2 \underline{y}\cdot Q\underline{y}dt \tag{8.18}$$

with

$$G = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} , \quad Q = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} .$$

Since this system is deterministic, the matrix B is identically zero so that Ψ_o from equation (7.14) is zero. The functional $\Psi(\underline{z},\tau)$ then has the form

$$\Psi(\underline{z},\tau) = \Psi_2(\tau)\underline{z}\cdot\underline{z} \tag{8.19}$$

where

$$\Psi_2(\tau) = \begin{vmatrix} \psi_{11} & \psi_{12} \\ \psi_{12} & \psi_{22} \end{vmatrix} .$$

Using equation (7.9), the optimum plant input $\underline{y}_{min}$ is determined as

$$\underline{y}_{min} = -\frac{1}{2\lambda^2}C^T\nabla_{\underline{z}}\Psi = -\frac{1}{\lambda^2}\begin{vmatrix} 0 & 0 \\ 0 & 1/\alpha \end{vmatrix}\begin{vmatrix} \psi_{11} & \psi_{12} \\ \psi_{12} & \psi_{22} \end{vmatrix}\begin{vmatrix} z_1 \\ z_2 \end{vmatrix} . \tag{8.20}$$

Expanding equation (8.6) results in

$$y_{i_{min}} = -\frac{1}{\alpha\lambda^2}[\psi_{12}z_1 + \psi_{22}z_2] . \tag{8.21}$$

The structure of the synthesized system thus assumes the form shown in Figure (2). In Figure (2), Laplace transform notation has been used to denote transfer functions and the blocks containing the ψ_{ij} denote time varying gain functions.

Using equation (7.16), the elements of the matrix Ψ_2 are determined by the solutions to the set of differential equations

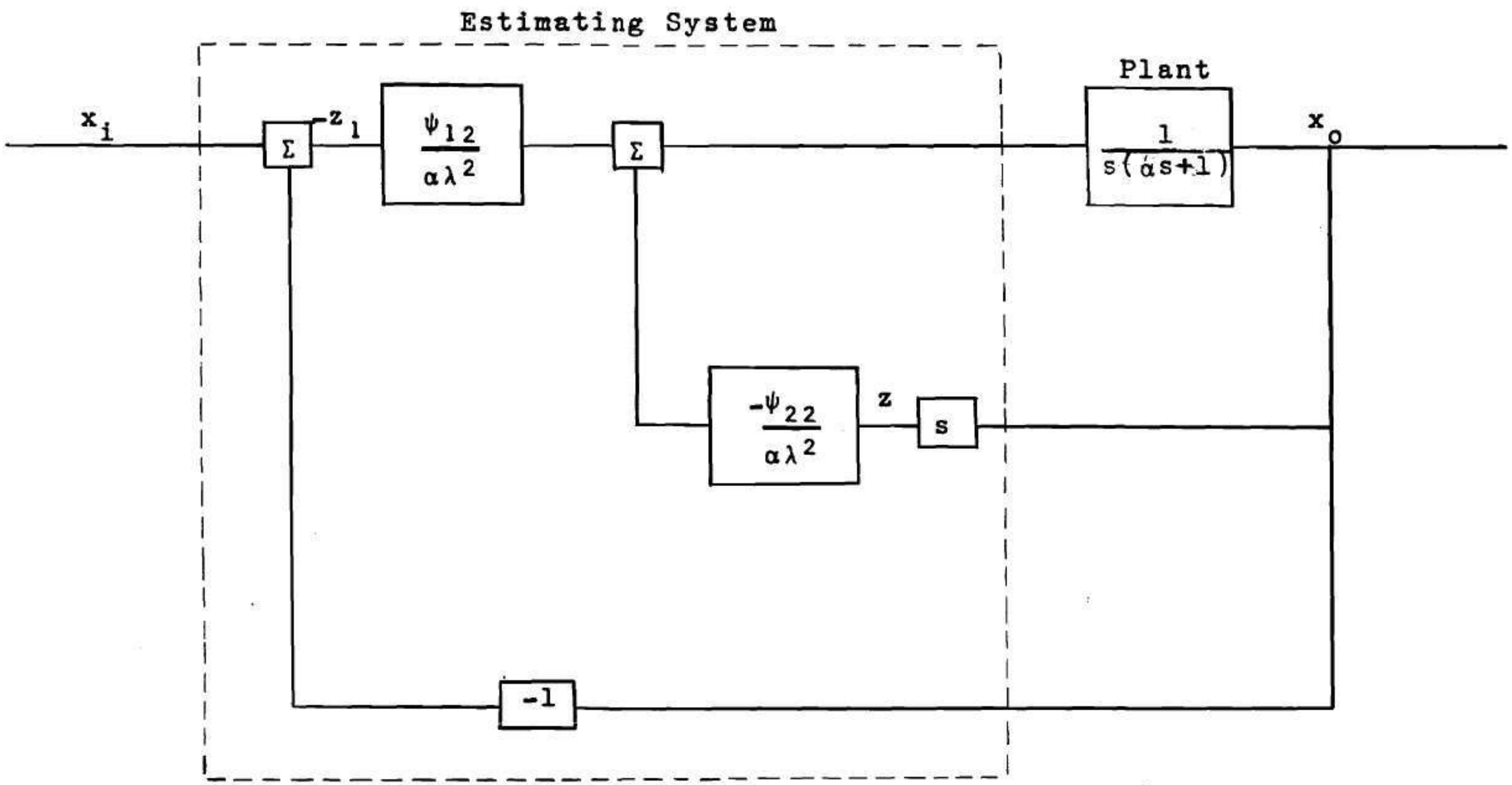

Figure 2. System Configuration for Example Three,
Cases One and Two.

$$\dot{\psi}_{11} = 1 - \frac{\psi_{12}^2}{\alpha^2\lambda^2}$$

$$\dot{\psi}_{22} = - \frac{\psi_{22}^2}{\alpha^2\lambda^2} - 2\psi_{12} - \frac{2}{\alpha}\psi_{22}$$

$$\dot{\psi}_{12} = - \frac{\psi_{12}\psi_{22}}{\alpha^2\lambda^2} + \psi_{11} - \frac{\psi_{12}}{\alpha}$$

with $\psi_{11}(0) = \psi_{12}(0) = \psi_{22}(0) = 0$.

Solutions for this set of equations for $\alpha = 1$, $\lambda = 1/4$ have been obtained and appear in Figure (3). The functions illustrated in Figure (3) determine the time varying gain parameters in Figure (2). Thus the synthesis procedure is completed. For T large the ψ_{ij} are constants so that if the measurement interval is long the system can be realized with time invariant gain parameters. In this case, the overall system is governed by a second order time invariant differential equation. For $\alpha = 1$, $\lambda = 1/4$, this second order system has a natural frequency of 2 rad/sec. and a damping ratio of 0.71. These values are consistent with results which would have been obtained by conventional design techniques.

The response characteristics of the synthesized system for several values of T appear in Figure (4). For shorter measurement intervals, the response characteristics for the time variable system are superior to those obtained with the constant parameter system.

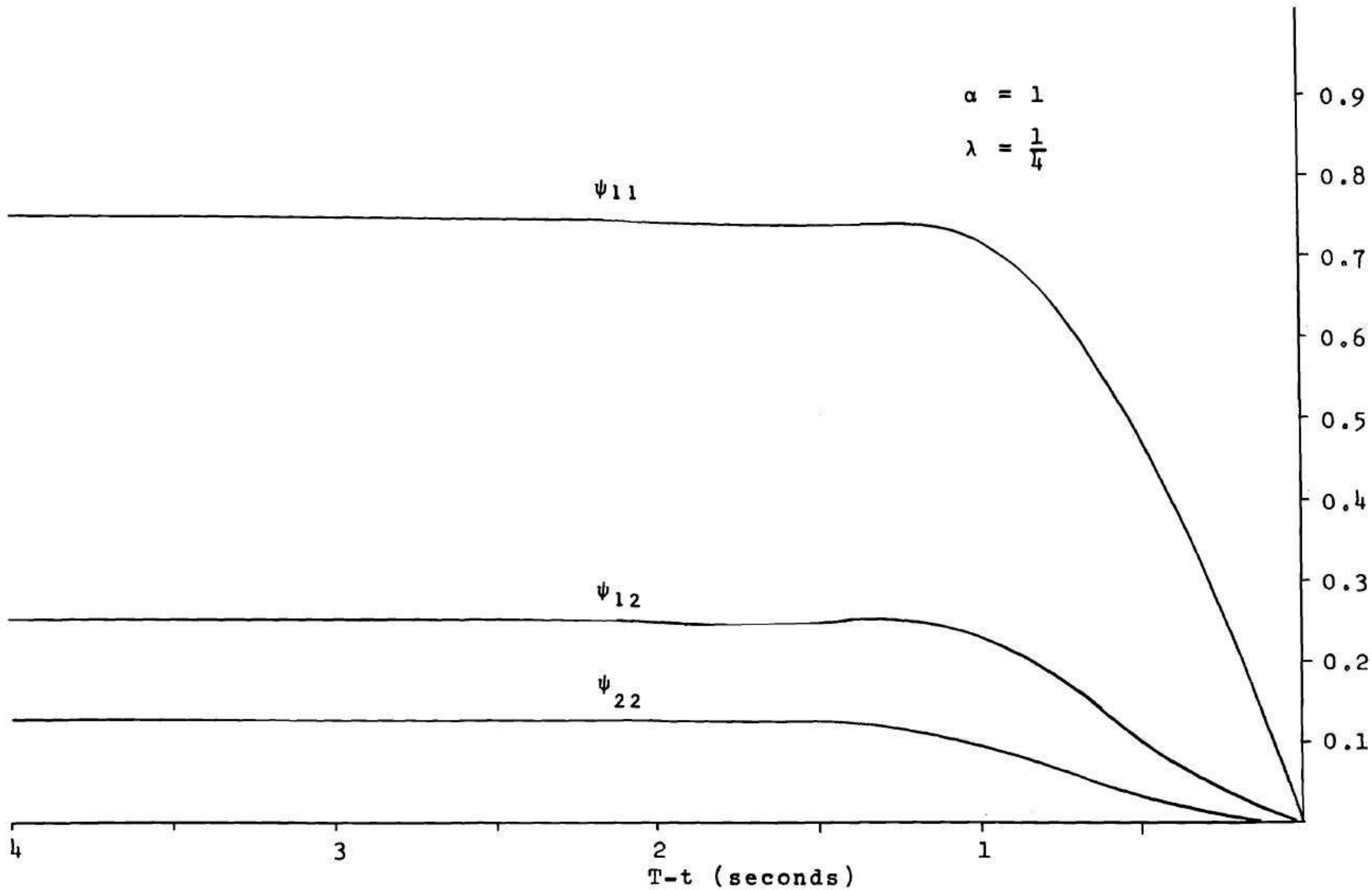

Figure 3. System Gains. Example Three, Case One.

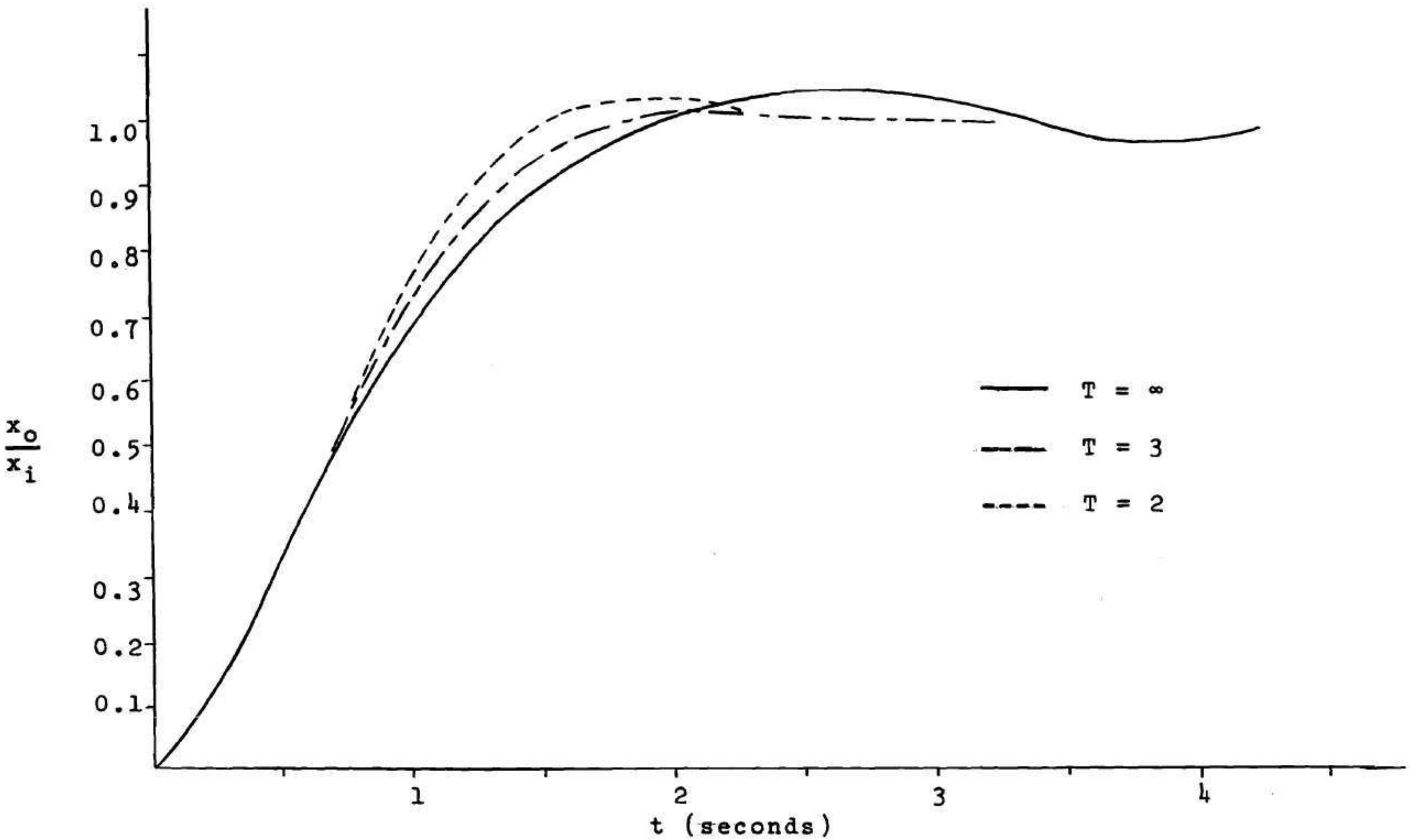

Figure 4. System Response. Example Three, Case One.

<u>Case Two. A Constant Input with a Final Value Performance Index</u>

Consider again the system defined by equation (8.16). Let it be required to measure the constant input only at time t = T. The performance index is taken as

$$J(t) = x_1^2(T) + \int_t^T \lambda^2 y_i^2 dt = \underline{z}\cdot G\underline{z} + \int_t^T \lambda^2 \underline{y}\cdot Q\underline{y} dt \quad . \tag{8.21}$$

Again the functional $\Psi(\underline{z},\tau)$ is of the form

$$\Psi(\underline{z},\tau) = \Psi_2(\tau)\underline{z}\cdot\underline{z}$$

and using equation (7.25) the plant input $y_i(t)$ is determined as

$$y_i = -\frac{1}{\alpha\lambda^2}[\psi_{12}z_1 + \psi_{22}z_i] \quad .$$

The structure of the synthesized system is thus given in Figure (2). The elements of the matrix $\Psi_2(\tau)$ for this system are determined from equation (7.24) as the solution of the differential equations

$$\dot{\psi}_{11} = -\frac{\psi_{12}^2}{\alpha^2\lambda^2}$$

$$\dot{\psi}_{22} = -\frac{\psi_{22}^2}{\alpha^2\lambda^2} + 2\psi_{12} - \frac{2}{\alpha}\psi_{22}$$

$$\dot{\psi}_{12} = -\frac{\psi_{12}\psi_{22}}{\alpha^2\lambda^2} + \psi_{11} - \frac{\psi_{12}}{\alpha}$$

with $\psi_{11}(0) = 1$, $\psi_{12}(0) = \psi_{22} = (0) = 0$. The solutions to these equations are given in Figure (5). These gain functions specify the final value system parameters. The response

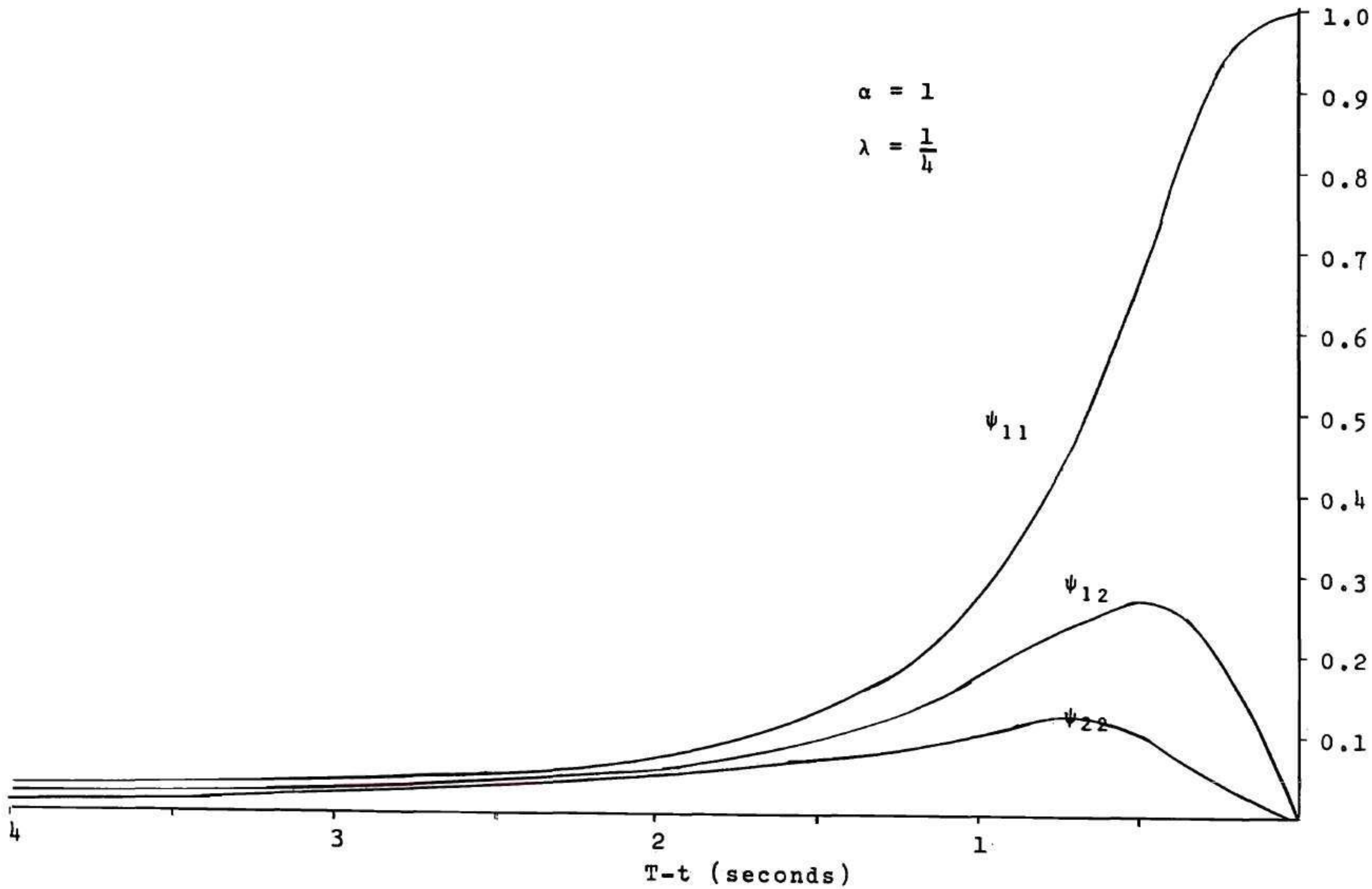

Figure 5. System Gains. Example Three, Case Two.

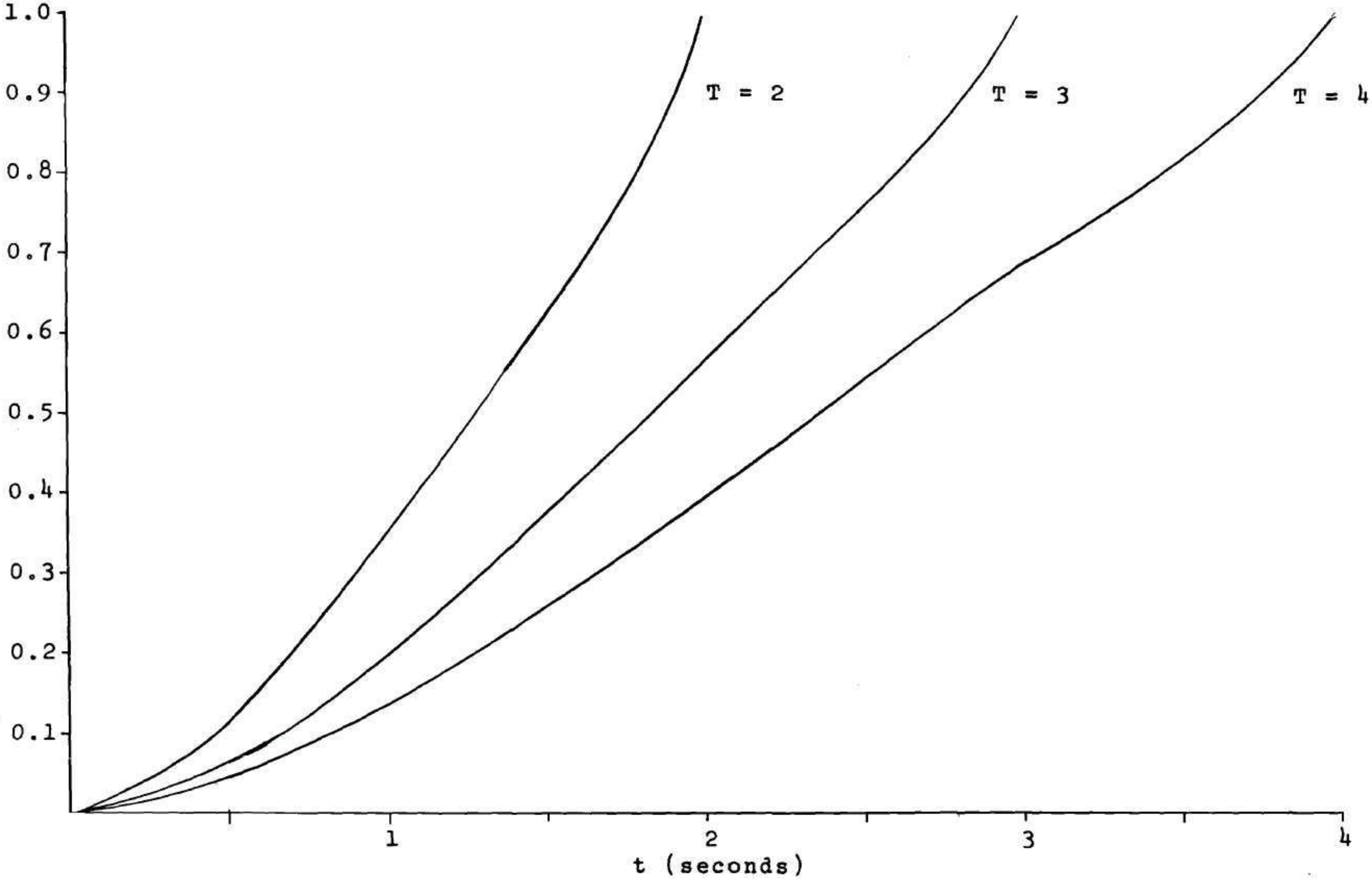

Figure 6. System Response. Example Three, Case Two.

characteristics for the final value systems are shown in Figure (6). As expected, the final value system functions in such a manner that the error between system input and output is large except near the specified time T.

Case Three. A Minimum Squared Error System with a Time Varying Input

The purpose of this example is to illustrate the synthesis procedure in the more general of a deterministic system with a time variable input.

Let the system plant again be specified by equation (8.15). Consider a decaying exponential input specified by $x_i(t) = e^{-\beta t}$. Then $\dot{x}_i(t) = -\beta e^{-\beta t}$ with $x_i(0) = 1$. By letting

$$z_1 = x_i$$

$$\dot{z}_1 = -\beta z_1$$

$$z_2 = x_o$$

$$\dot{z}_2 = z_3$$

$$\dot{z}_3 = \frac{-z_3}{\alpha} + \frac{y_i}{\alpha}$$

the system mathematical model can be expressed as

$$\underline{\dot{z}} = A\underline{z} + C\underline{y} \qquad (8.22)$$

with

$$\underline{z} = \begin{vmatrix} z_1 \\ z_2 \\ z_3 \end{vmatrix}, \qquad \underline{y} = \begin{vmatrix} 0 \\ 0 \\ y_i \end{vmatrix}$$

$$A = \begin{vmatrix} -\beta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\alpha \end{vmatrix}, \quad C = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\alpha \end{vmatrix} .$$

The system performance index is taken as

$$J(t) = \int_t^T [(x_i - x_o)^2 + \lambda^2 y_i^2]dt = \int_t^T [\underline{z}\cdot G\underline{z} + \lambda^2 \underline{y}\cdot Q\underline{y}]dt \tag{8.23}$$

where

$$G = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad Q = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} .$$

The functional $\Psi(\underline{z},\tau)$ is again of the form

$$\Psi(\underline{z},\tau) = \Psi_2(\tau)\underline{z}\cdot\underline{z} \tag{8.24}$$

with $\Psi_2(\tau)$ given by

$$\Psi_2(\tau) = \begin{vmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{12} & \psi_{22} & \psi_{23} \\ \psi_{13} & \psi_{23} & \psi_{33} \end{vmatrix} .$$

Using equation (7.9), the plant input is determined as

$$\underline{y}_{min} = -\frac{1}{\alpha\lambda^2}\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\begin{vmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{12} & \psi_{22} & \psi_{23} \\ \psi_{13} & \psi_{23} & \psi_{33} \end{vmatrix}\begin{vmatrix} z_1 \\ z_2 \\ z_3 \end{vmatrix}$$

which when expanded yields

$$y_{i_{min}} = -\frac{1}{\alpha\lambda^2}[\psi_{13}z_1 + \psi_{23}z_2 + \psi_{33}z_3] \quad . \tag{8.25}$$

The structure of the synthesized system is then given in Figure (7).

Then, using equation (7.16) the elements of the matrix $\Psi_2(\tau)$ are determined by the solutions to the differential equations

$$\dot{\psi}_{11} = 1 - \frac{\psi_{13}^2}{\alpha^2\lambda^2} - 2\beta\psi_{11}$$

$$\dot{\psi}_{22} = 1 - \frac{\psi_{23}^2}{\alpha^2\lambda^2}$$

$$\dot{\psi}_{33} = -\frac{\psi_{33}^2}{\alpha^2\lambda^2} + 2\psi_{23} - \frac{2}{\alpha}\psi_{33}$$

$$\dot{\psi}_{12} = -1 - \frac{\psi_{13}\psi_{23}}{\alpha^2\lambda^2} - \beta\psi_{12}$$

$$\dot{\psi}_{13} = -\frac{\psi_{13}\psi_{33}}{\alpha^2\lambda^2} - \beta\psi_{13} + \psi_{12} - \frac{\psi_{13}}{\alpha}$$

$$\dot{\psi}_{23} = -\frac{\psi_{23}\psi_{33}}{\alpha^2\lambda^2} + \psi_{22} - \psi_{23}$$

with $\psi_{ij}(0) = 0$, for i, j = 1,2,3.

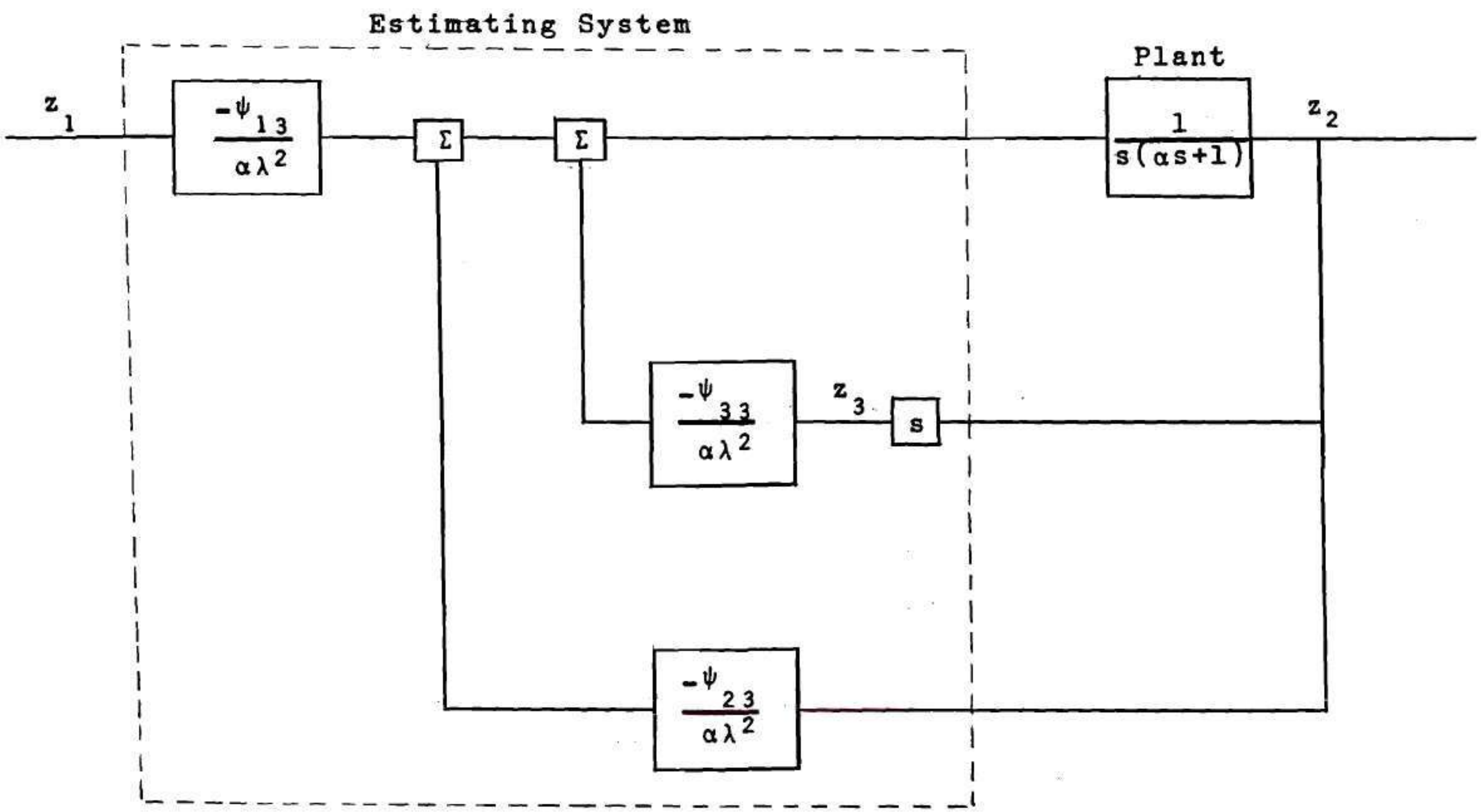

Figure 7. System Configuration, Example Three, Case Three.

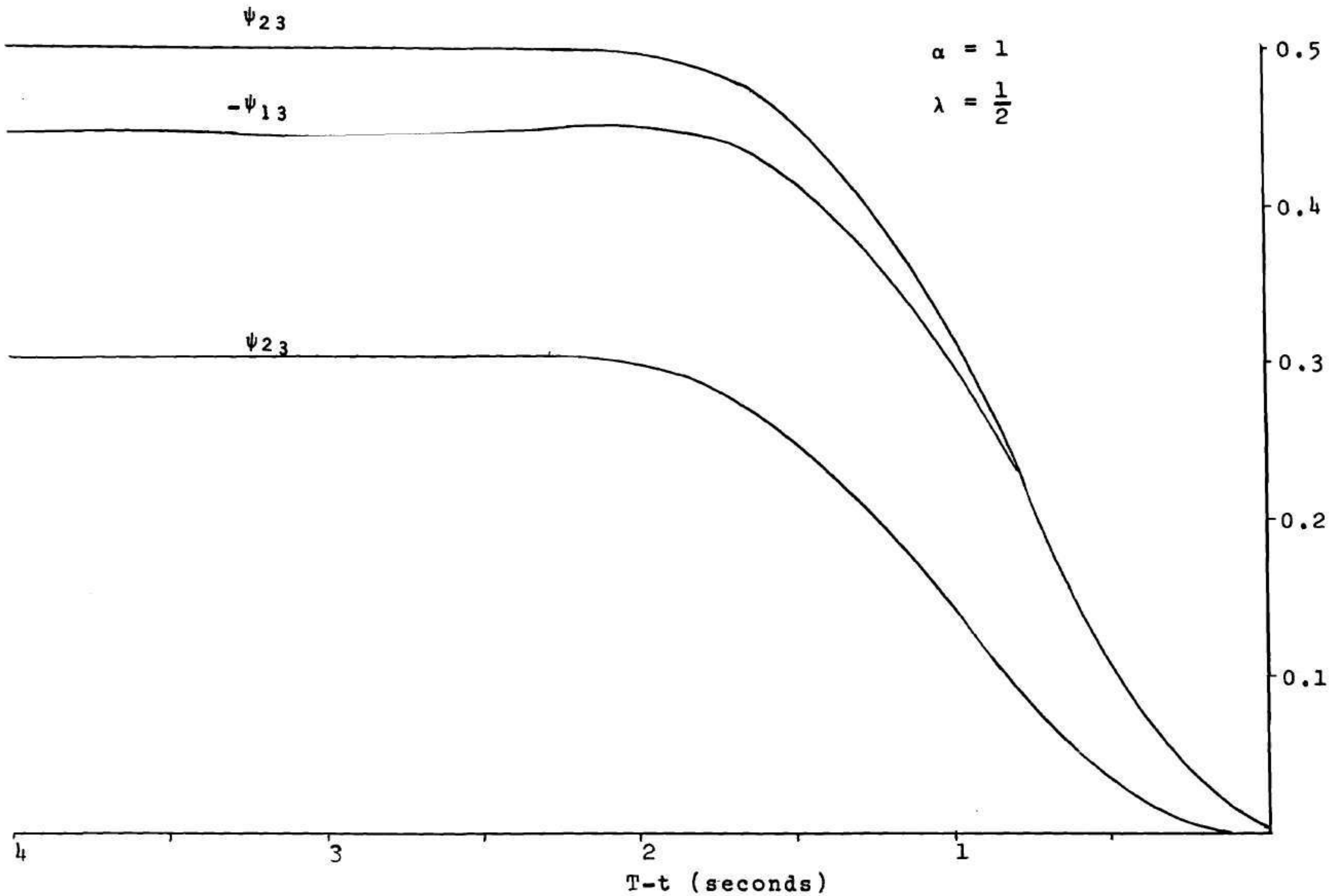

Figure 8. System Gains. Example Three, Case Three.

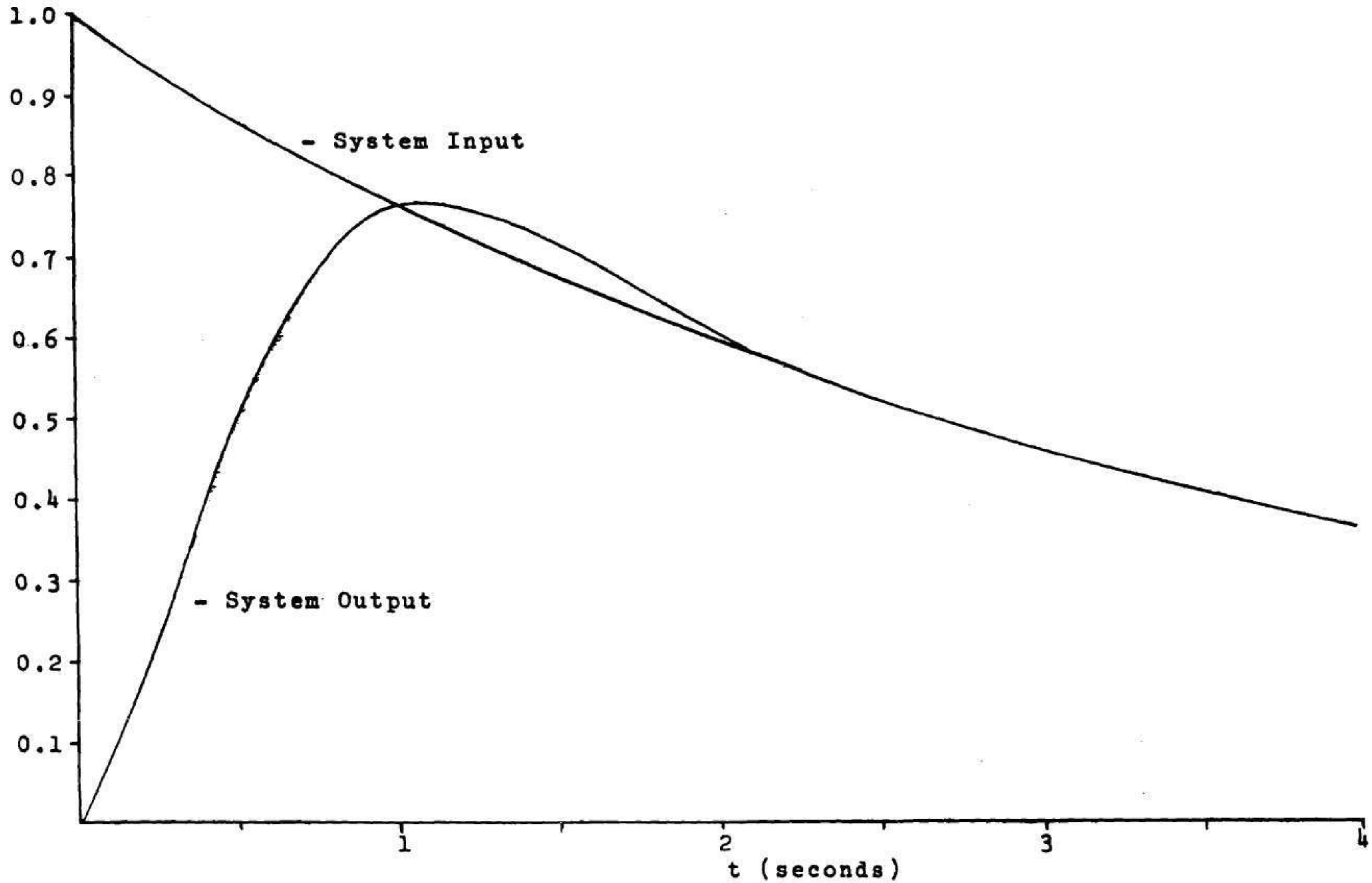

Figure 9. System Response. Example Three, Case Three.

The solution to these equations with $\alpha = 1$, $\lambda = 1/4$, $\beta = 1/4$ are shown in Figure (8). The response of the synthesized system is shown in Figure (9).

Example Four. System with Random Disturbance

Consider the problem of guiding an aircraft along a specified trajectory. Let γ_i denote the required flight path angle and γ_o the actual flight path angle of the aircraft. The aircraft is actuated by the control surface deflection δ. It is assumed that for fixed δ the aircraft flies in a circular arc of radius ρ which is inversely proportional to δ. That is $\rho = K_1/\delta$. The aircraft forward velocity is assumed constant so that $\rho\dot{\gamma} = V_o$ where V_o is the velocity. Then the relation between γ and δ is $\dot{\gamma} = K_1\delta$ where $K_1 = K/V_o$. The dynamics of the control surface actuator are assumed to satisfy the differential equation

$$\alpha\ddot{\delta} + \dot{\delta} = y_i$$

where y_i is the actuator input to be determined.

The effects of inaccuracy is measuring γ_o and the effects of turbulence are represented by a disturbance signal $n(t)$ which adds to the actuator output δ. It is assumed that $n(t)$ has zero mean and covariance $r_{nn}(t_1,t_2) = e^{-\beta|t_1,-t_2|}$. Using the results of Appendix III, the disturbance $n(t)$ satisfies the differential equation

$$\dot{n}(t) = -\beta n(t) + \dot{u} \quad .$$

Then assuming that K = 1 and for simplicity that $\gamma_i(t) = 1$ and identifying

$$z_1 = \gamma_o - 1$$

$$z_2 = \delta$$

$$z_4 = n(t)$$

the mathematical system model becomes

$$\dot{z}_1 = z_2 + z_4$$

$$\dot{z}_2 = z_3$$

$$\dot{z}_3 = -\frac{z_3}{\alpha} + \frac{y_i}{\alpha}$$

$$\dot{z}_4 = -\beta z_4 + \dot{u} \quad .$$

In vector notation these equations can be written as

$$\dot{\underline{z}} = A\underline{z} + B\dot{u} + C\underline{y} \tag{8.26}$$

with

$$\underline{z} = \begin{vmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{vmatrix} , \quad y = \begin{vmatrix} 0 \\ 0 \\ y_i \\ 0 \end{vmatrix} , \quad B = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} , \quad A = \begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/\alpha & 0 \\ 0 & 0 & 0 & -\beta \end{vmatrix} \quad .$$

The system is required to minimize the expected value of the error between γ_o and γ_i. Structural consideration might require a constraint on the rate of change $\dot{\gamma}$. It is also assumed that the input energy to the actuator is limited.

These requirements can be expressed mathematically by the performance index

$$J(t) = \int_t^T (z_1^2 + \lambda_1^2 z_2^2 + \lambda_2^2 y_i^2)dt \qquad (8.27)$$

or in vector notation

$$J(t) = \int_t^T (\underline{z}\cdot G\underline{z} + \lambda_2^2 \underline{y}\cdot Q\underline{y})dt$$

where

$$G = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad Q = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} .$$

The functional $\Psi(\underline{z},\tau)$ which minimizes the expected value of equation (8.27) is of the form

$$\Psi(\underline{z},\tau) = \Psi_o(\tau) + \Psi_2(\tau)\underline{z}\cdot\underline{z} \qquad (8.28)$$

where Ψ_o is a scalar and $\Psi_2(\tau)$ is given by

$$\Psi_2(\tau) = \begin{vmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{12} & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_{13} & \psi_{23} & \psi_{33} & \psi_{34} \\ \psi_{14} & \psi_{24} & \psi_{34} & \psi_{44} \end{vmatrix} .$$

Using equation (7.9) the $y_{i_{min}}$ is determined as

$$y_{i_{min}} = - \frac{1}{\alpha\lambda_2^2}[\psi_{13}z_1 + \psi_{23}z_2 + \psi_{33}z_3 + \psi_{34}z_4] . \qquad (8.29)$$

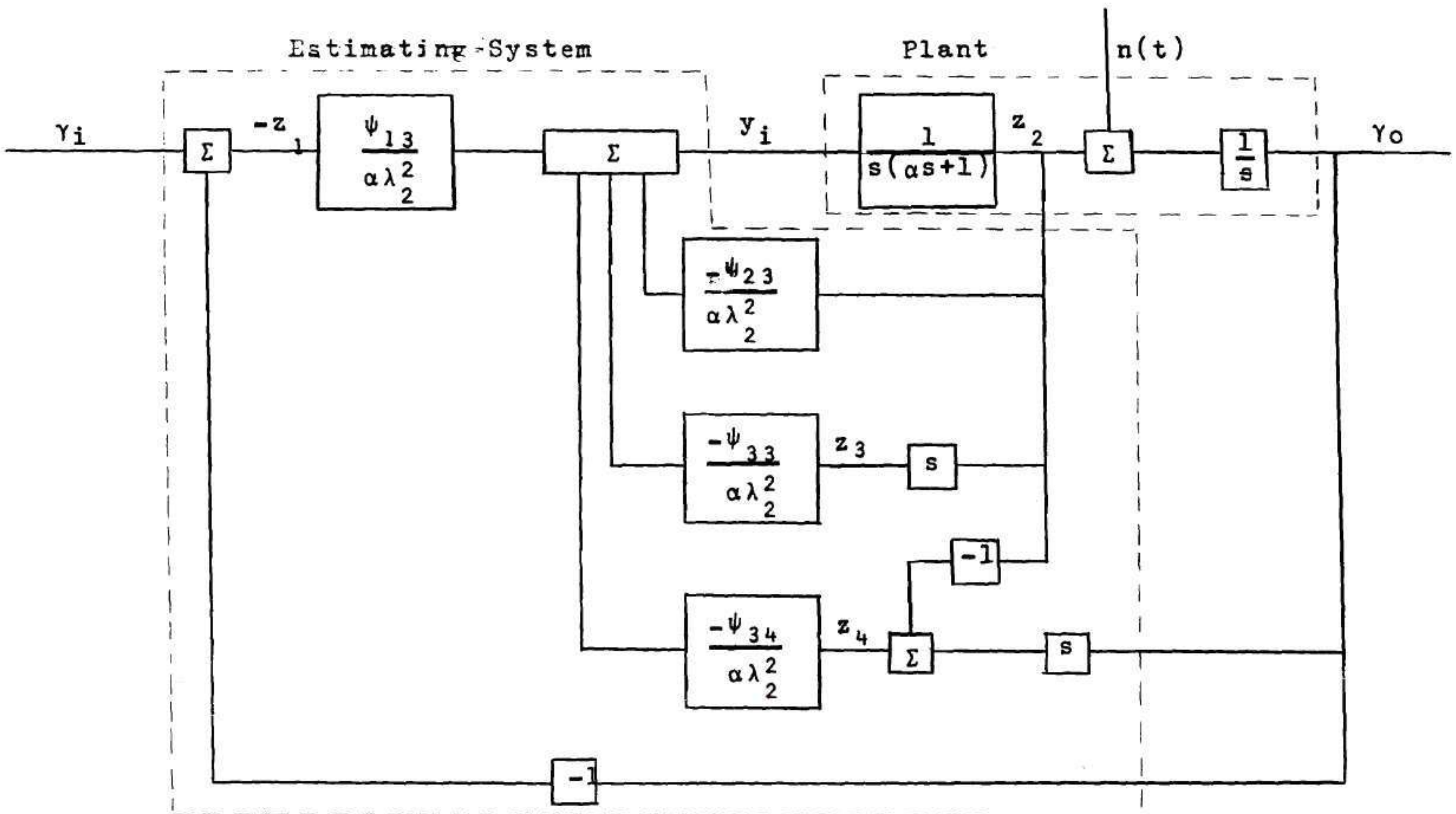

Figure 10. System Configuration. Example Four.

Using equations (7.14), (7.16) the scalar $\Psi_o(\tau)$ and the elements of the matrix $\Psi_2(\tau)$ are determined by the solutions to the differential equations

$$\dot{\Psi}_o = \frac{1}{2}\psi_{44}$$

$$\dot{\psi}_{11} = 1 - \frac{\psi_{13}^2}{\alpha^2\lambda_2^2}$$

$$\dot{\psi}_{22} = \frac{1}{\lambda^2}\frac{\psi_{23}^2}{\alpha^2\lambda_2^2} + 2\psi_{12}$$

$$\dot{\psi}_{33} = -\frac{\psi_{33}^2}{\alpha^2\lambda_2^2} + 2\psi_{23} - \frac{2\psi_{33}}{\alpha}$$

$$\dot{\psi}_{44} = -\frac{\psi_{34}^2}{\alpha^2\lambda_2^2} + 2\psi_{14} - 2\beta\psi_{44}$$

$$\dot{\psi}_{12} = -\frac{1}{\alpha^2\lambda_2^2}\psi_{13}\psi_{23} + \psi_{11}$$

$$\dot{\psi}_{13} = -\frac{1}{\alpha^2\lambda_2^2}\psi_{33}\psi_{13} + \psi_{12} - \frac{1}{\alpha}\psi_{13}$$

$$\dot{\psi}_{14} = -\frac{1}{\alpha^2\lambda_2^2}\psi_{13}\psi_{34} - \frac{1}{\alpha}\psi_{11} - \beta\psi_{14}$$

$$\dot{\psi}_{23} = -\frac{1}{\alpha^2\lambda_2^2}\psi_{23}\psi_{33} + \psi_{13} + \psi_{22} - \frac{1}{\alpha}\psi_{23}$$

$$\dot{\psi}_{24} = -\frac{1}{\alpha^2\lambda_2^2}\psi_{23}\psi_{34} + \psi_{14} + \psi_{12} - \beta\psi_{24}$$

$$\dot{\psi}_{34} = -\frac{1}{\alpha^2\lambda_2^2}\psi_{33}\psi_{34} + \psi_{24} - \frac{1}{\alpha}\psi_{34} + \psi_{13} - \beta\psi_{34}$$

with $\Psi_o(0) = 0$, $\psi_{ij}(0) = 0$, $i,j = 1,\cdots, 4$.

Numerical solutions to these equations have been obtained for $\alpha = 1$, $\lambda_1 = 1$, $\lambda_2 = 1/4$. Figure (11) shows the functions necessary to specify the parameters in Figure (10).

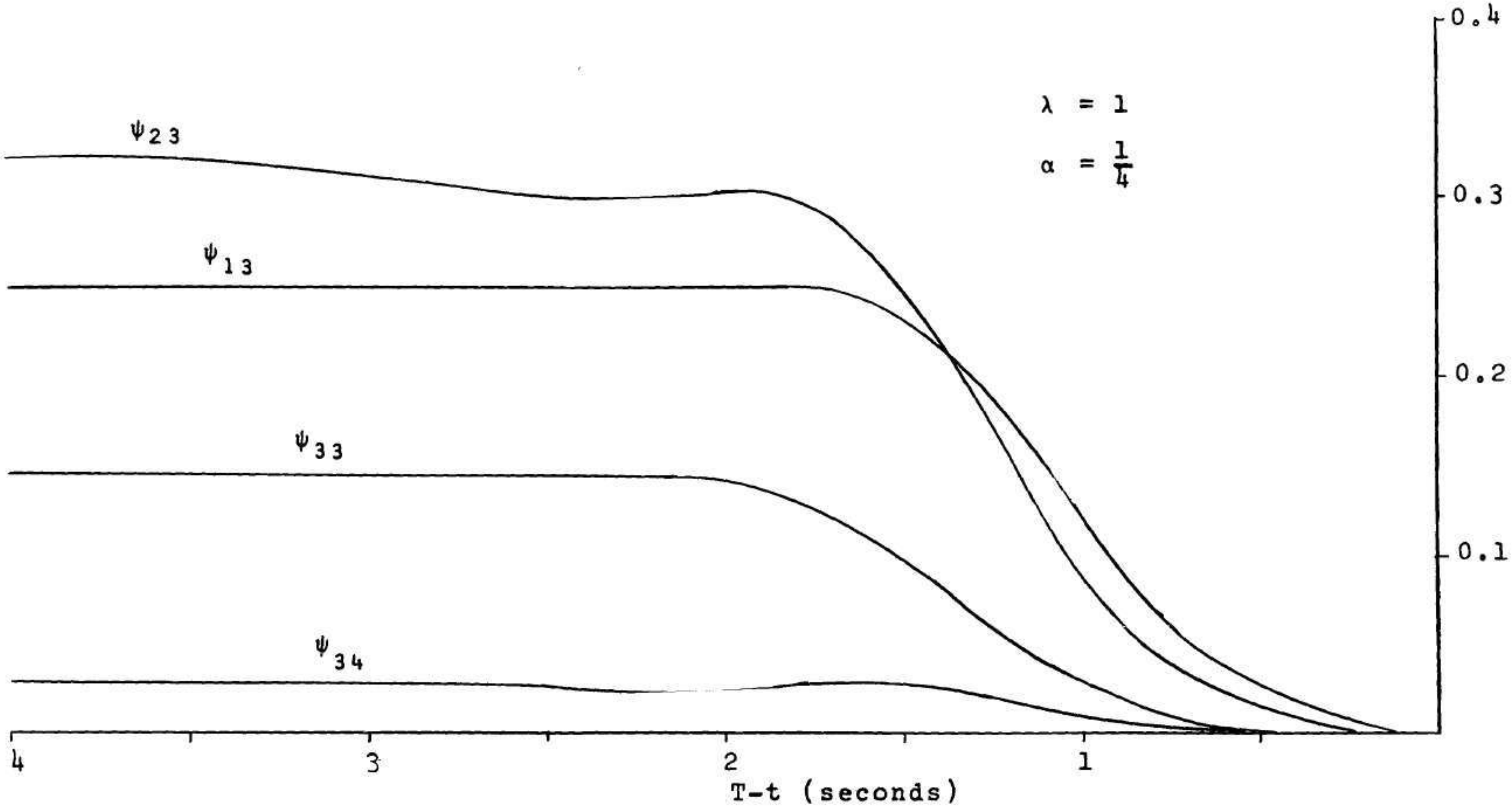

Figure 11. System Gains. Example Four

Example Five. A System With a Random Input

Consider a system with an input consisting of the random process discussed in example two. Let the system plant dynamics be specified by the differential equation (8.15). Making the identification

$$z_1 = w$$

$$z_2 = x_o$$

the system mathematical model becomes

$$\dot{\underline{z}} = A\underline{z} + B\dot{u} + C\underline{y} \tag{8.30}$$

where

$$\underline{z} = \begin{vmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{vmatrix}, \quad \underline{y} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ y_i \end{vmatrix}, \quad B = \begin{vmatrix} \sqrt{7}/2t \\ \sqrt{12}/t^2 \\ 0 \\ 0 \end{vmatrix}$$

$$A = \begin{vmatrix} -4/t & 1 & 0 & 0 \\ 2/t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1/\alpha \end{vmatrix}, \quad C = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\alpha \end{vmatrix}.$$

The performance index is chosen as

$$J(t) = \int_t^T (z_1 - z_3)^2 + \lambda^2 y_i^2 dt = \int_t^T \underline{z}\cdot G\underline{z} + \lambda^2 \underline{y}\cdot Q\underline{y} dt$$

where

$$G = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad Q = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} .$$

The minimizing functional $\Psi(\underline{z},\tau)$ is again of the form

$$\Psi(\underline{z},\tau) = \Psi_o(\tau) + \Psi_2(\tau)\underline{z}\cdot\underline{z}$$

where Ψ_o is a scalar and the matrix Ψ_2 is given by

$$\Psi_2(\tau) = \begin{vmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{12} & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_{13} & \psi_{23} & \psi_{33} & \psi_{34} \\ \psi_{14} & \psi_{24} & \psi_{34} & \psi_{44} \end{vmatrix} .$$

The plant input is determined from equation (7.9) as

$$y_{i_{min}} = -\frac{1}{\alpha\lambda^2}[\psi_{14}z_1 + \psi_{24}z_2 + \psi_{34}z_3 + \psi_{44}z_4]. \quad (8.31)$$

The synthesized system structure in this case is shown in Figure (12).

The scalar $\Psi_o(\tau)$ and the elements of the matrix $\Psi_2(\tau)$ are determined from equations (7.14), (7.16) as the solutions to the differential equations

$$\dot{\Psi}_o = \frac{1}{2}\left[\psi_{11}\frac{7}{4(T-\tau)^2} + \psi_{22}\frac{12}{(T-\tau)^4}\right]$$

$$\dot{\psi}_{11} = 1 - \frac{\psi_{14}^2}{\alpha^2\lambda^2} - \frac{8}{(T-\tau)}\psi_{11} + \frac{4}{(T-\tau)^2}\psi_{12}$$

$$\dot{\psi}_{22} = -\frac{\psi_{24}^2}{\alpha^2\lambda^2} + 2\psi_{12}$$

$$\dot{\psi}_{33} = 1 - \frac{\psi_{34}^2}{\alpha^2\lambda^2}$$

$$\dot{\psi}_{44} = -\frac{\psi_{44}^2}{\alpha^2\lambda^2} + 2\psi_{34} - \frac{2}{\alpha}\psi_{44}$$

$$\dot{\psi}_{12} = -\frac{\psi_{14}\psi_{24}}{\alpha^2\lambda^2} - \frac{4}{(T-\tau)}\psi_{12} + \frac{2\psi_{22}}{(T-\tau)^2} + \psi_{11}$$

$$\dot{\psi}_{13} = -1 - \frac{\psi_{14}\psi_{34}}{\alpha^2\lambda^2} - \frac{4}{T-\tau}\psi_{13} + \frac{2}{(T-\tau)^2}\psi_{23}$$

$$\dot{\psi}_{14} = -\frac{\psi_{14}\psi_{44}}{\alpha^2\lambda^2} - \frac{4}{T-\tau}\psi_{14} + \frac{2}{(T-\tau)^2}\psi_{24} + \psi_{13} - \frac{\psi_{14}}{\alpha}$$

$$\dot{\psi}_{23} = -\frac{\psi_{24}\psi_{34}}{\alpha^2\lambda^2} + \psi_{13}$$

$$\dot{\psi}_{24} = -\frac{\psi_{24}\psi_{44}}{\alpha^2\lambda^2} + \psi_{14} + \psi_{23} - \frac{1}{\alpha}\psi_{24}$$

$$\dot{\psi}_{34} = -\frac{\psi_{34}\psi_{44}}{\alpha^2\lambda^2} + \psi_{33} - \frac{\psi_{34}}{\alpha}$$

with $\Psi_o(0) = 0$, $\psi_{ij}(0) = 0$, $i,j = 1,\cdots, 4$.

The numerical solutions to these equations for $\alpha = 1$, $\lambda = 1/4$, $T = 4$ have been obtained. The functions necessary to specify the parameters in Figure (12) are shown in Figure (13).

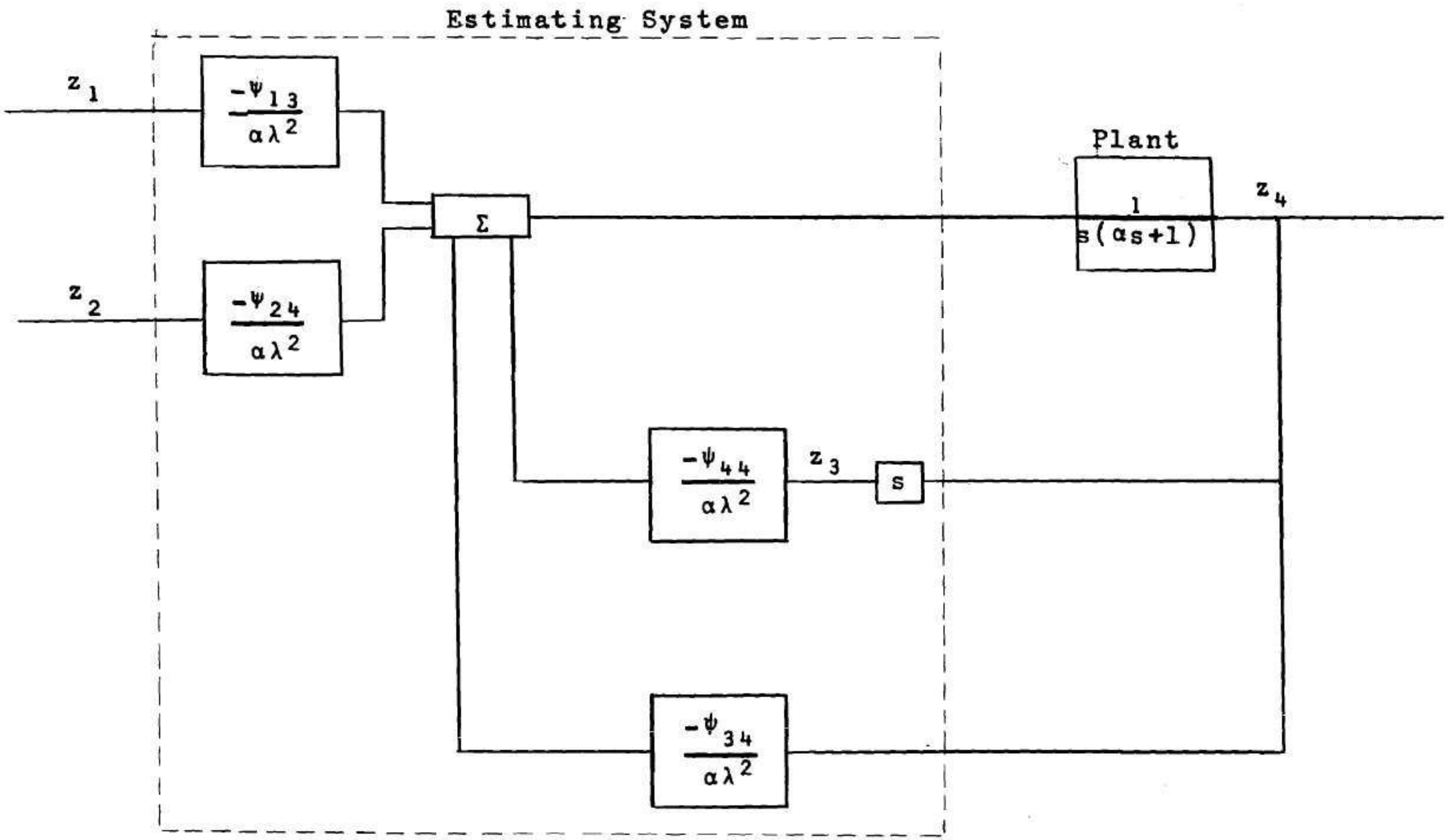

Figure 12. System Configuration. Example Five.

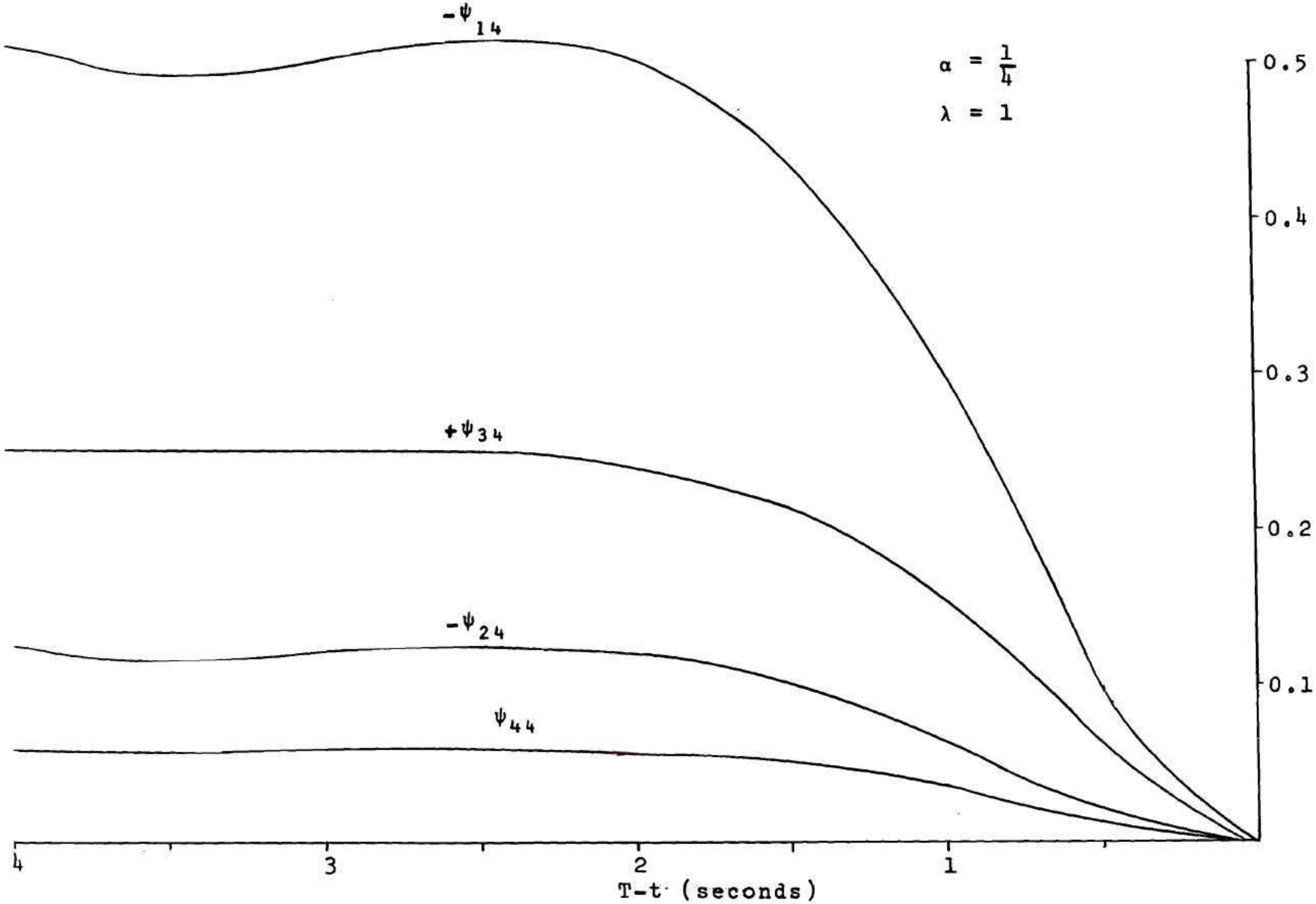

Figure 13. System Gains. Example Five.

CHAPTER IX

CONCLUSIONS

In the preceding chapters, a synthesis method applicable to the design of a class of measurement systems has been presented. The method can be applied to the design of systems with either random or nonrandom input and disturbance signals. Multivariate systems may be treated and several design constraints can be accommodated simultaneously. Although only systems with linear plants have been considered in this presentation, the modifications required to accommodate nonlinear system elements are at least conceptually quite simple.

It has been shown that system stability requirements can be established directly from the solution of the functional equation. The stability conditions reduce to the single requirement that $\Psi(\underline{z},t)$ is bounded when the measurement interval is infinite.

Application of the synthesis method is straightforward. In most cases, either an analog or digital computer is required to implement the required computations. In general, the synthesized system takes the form of a feedback system with time-varying feedback gains. Consequently, time variable elements are required in the system realization. In the special case of stationary systems whose inputs vary slowly with respect to the plant time constants, the computational procedures are simplified and the resulting system can be realized with stationary elements.

In order to carry out the synthesis procedure, the following must be specified:

1. The characteristics of the plant to be employed.

2. The characteristics of the input and disturbance signals, at least through the second order statistical moments.

3. A set of design requirements and constraints.

With these specifications, the synthesis procedure involves the following steps.

Step 1. A mathematical model for the measurement system is formulated in the form of equation (3.9). This involves determining the differential equations which describe the dynamics of the plant, the inputs, and the disturbances. The state variable representation used in equation (3.9) is discussed in Appendix 1. When the input and disturbance signals are stochastic, these signals must be represented in the form of equation (3.2) before equation (3.9) can be formulated. After writing the specified covariance function for each random signal in the form of equation (A-3.6), the elements of the matrices $A^w(t)$ and $B(t)$ in equation (3.2) can be determined using equations (A-1.5), (A-1.15), and (A-3.16).

Step 2. The system design requirements and constraints are stated mathematically in terms of a performance index of the form of equation (4.4). This involves defining an appropriate system error functional which is a measure of the deviation of the system performance from the ideal. In the cases where either a minimum squared error or a final value performance index is appropriate, the performance index can be written in the form of either equation (7.6) or equation (7.18). In these equations, G and Q are symmetric matrices which are chosen in accordance with the specified design requirements and λ is an arbitrary design parameter.

Step 3. The chosen error functional is substituted into the functional equation (5.10). The minimization operation is then performed, thus determining the plant input vector as a function of the minimum of the expected value of the performance index. In the case of either of the two specific performance indices mentioned above, this operation results in an expression in the form of equation (7.9). The operator $\nabla_{\underline{z}}$ is defined after equation (5.4).

Step 4. After eliminating the plant input vector from equation (5.10) by means of the minimization operation, the functional equation is solved. To solve this equation, the functional $\Psi(\underline{z},\tau)$ is expanded in a series such as equation (7.11). Substitution of this series into the functional equation results in a set of ordinary differential equations which are solved to determine the functional, $\Psi(\underline{z},\tau)$. For the minimum squared error performance indices, this set of differential equations is specified explicitly by equations (7.14 - 7.16). For the final value performance index, the differential equations take the form of equations (7.22 - 7.24).

Completion of these steps defines the structure and parameters of the synthesized system and completes the synthesis procedure. The examples presented in Chapter VIII illustrate the procedure outlined above.

APPENDIX I

THE REPRESENTATION OF LINEAR DYNAMIC SYSTEMS

This appendix is included to bring together in one place material relevant to the representation of linear dynamic systems by ordinary differential equations. In particular, the so-called "state space" or vector differential equation representation is discussed and related to the more conventional n^{th} order differential equation representation. Although this material is well known, no single reference presents it in the form required here. The majority of the material presented in this appendix can be found in references (16, 17, 20).

Representation By n^{th} Order Differential Equations

Consider a system representation in terms of the n^{th} order differential equation

$$L_t x(t) = N_t y(t) \qquad \text{(A-1.1)}$$

$$x(0) = 0$$

where x(t) is the system output variable and y(t) the system input. In equation (A-1.1) the operators L_t and N_t are defined by

$$L_t = \sum_{\nu=o}^{n} p_\nu(t) \frac{d^\nu}{dt^\nu}, \text{ with } p_n = 1 \qquad \text{(A-1.2)}$$

$$N_t = \sum_{\nu=o}^{m} q_\nu(t) \frac{d^\nu}{dt^\nu} \quad . \tag{A-1.3}$$

Associated with equation (A-1.1) is the homogeneous differential equation

$$L_t x(t) = 0 \tag{A-1.4}$$

$$x(0) = 0 \quad .$$

It is known* that if $\phi_1, \phi_2, \cdots, \phi_n$ are n linearly independent solutions of equation (A-1.4), then the operator L_t is specified by

$$L_t x(t) = W(x, \phi_1, \phi_2, \cdots, \phi_n)(t) = 0 \tag{A-1.5}$$

where

$$W(x, \phi_1, \cdots, \phi_n)(t) = \det \begin{vmatrix} x & \phi_1 & \cdot & \cdot & \phi_n \\ \dot{x} & \dot{\phi}_1 & \cdot & \cdot & \dot{\phi}_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^n & \phi^n & \cdot & \cdot & \phi_n^n \end{vmatrix} \quad .$$

In addition, for the homogeneous equation (A-1.4) there exists a function, H(t,s), defined by

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*<u>op</u>.<u>cit</u>., (20), pg. 83.

$$H(t,s) = \begin{cases} 0 & (s>t) \\ \dfrac{1}{W(\phi_1,\phi_2,\cdots,\phi_n)(s)} \begin{vmatrix} \phi_1(s) & \cdot & \cdot & \phi_n(s) \\ \dot{\phi}_1(s) & \cdot & \cdot & \dot{\phi}_n(s) \\ \cdot & \cdot & \cdot & \cdot \\ \phi_1(s) & \cdot & \cdot & \phi_n(s) \\ \phi_1(t) & & & \phi_n(t) \end{vmatrix} & (t \geq s) \end{cases} \qquad \text{(A-1.6)}$$

which satisfies

$$L_t H(t,s) = 0 \quad .$$

The nonhomogeneous differential equation (A-1.1) then has the unique solution

$$x(t) = \int_0^t H(t,s) N_s y(s) ds \quad . \qquad \text{(A-1.7)}$$

Using Green's Formula*, equation (A-1.7) can be written as

$$x(t) = \int_0^t G(t,s) y(s) ds \quad . \qquad \text{(A-1.8)}$$

where $G(t,s)$ satisfies $L_t[G(t,s)] = 0$. The function $G(t,s)$ is referred to as the Green's function for the equation (A-1.1). In network theory terminology, $G(t,s)$ would be called the impulse response function for a network described by equation (A-1.1).

The State Space Representation

It is frequently useful to describe a linear dynamic system by a linear vector differential equation of the form

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*op.cit. (2), pg. 86.

$$\dot{\underline{\tilde{x}}} = P(t)\underline{\tilde{x}} + Q(t)\underline{\tilde{y}} \qquad \text{(A-1.9)}$$

$$\underline{\tilde{x}}(0) = 0$$

rather than by n^{th} order scalar equations such as (A-1.1). Several differential equations in the form of equation (A-1.1) can be represented by a single vector equation in the form of equation (A-1.9) for appropriate choices of the vectors $\underline{\tilde{x}}$, $\underline{\tilde{y}}$. The vector $\underline{\tilde{x}}$ is referred to as the system state vector and the representation in terms of equations such as (A-1.9) is referred to as state space representation. The outputs of a system represented by equation (A-1.9) are described by a linear transformation of the state vector x(t). In particular, an equation of the form (A-1.1) can be converted to the form of equation (A-1.9). The conversion is not unique. For instance, an obvious way to convert from the form of equation (A-1.1) to that of equation (A-1.9) is to make the identification

$$x_1 = x$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\cdot$$

$$\cdot$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n = -p_o x_1 - p_1 x_2 \cdot \cdot \cdot -p_{n-1} x_n$$

$$y_1 = y$$

$$\dot{y}_1 = y_2$$

$$\cdot$$

$$\cdot$$

$$\dot{y}_{m-1} = y_m \quad .$$

Then the representation in terms of the vector equation (A-1.9) results with

$$\underline{x} = \begin{vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{vmatrix} \qquad y = \begin{vmatrix} y_1 \\ \cdot \\ \cdot \\ y_m \\ o \\ \cdot \\ o \end{vmatrix} \quad \text{(n-m+1 zeros)}$$

$$P = \begin{vmatrix} o & 1 & o & \cdot & \cdot & o \\ o & o & 1 & o & \cdot & o \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & o \\ o & & & & o & 1 \\ -p_o & -p & \cdot & \cdot & \cdot & -p_{n-1} \end{vmatrix} ,$$

$$Q = \begin{vmatrix} & & & 0 & & & \\ q_o & q_1 & \cdot & q_m & 1 & \cdot & 1 \end{vmatrix} .$$

(n-m+1 ones)

Associated with the vector differential equation (A-1.9) is the homogeneous differential equation

$$\dot{\underline{\tilde{x}}} = P(t)\underline{\tilde{x}} \qquad \text{(A-1.10)}$$

$$\underline{\tilde{x}}(0) = 0 \quad .$$

If ϕ_1, ϕ_2, $\cdots$, ϕ_n are n linear independent solutions to equation (A-1.4) for equation (A-1.10) there exists a fundamental matrix, $\Phi(t)$, defined by

$$\tilde{\Phi}(t) = \begin{vmatrix} \phi_1 & \phi_2 & \cdot & \cdot & \phi_n \\ \dot{\phi}_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_1^n & \phi_2^n & \cdot & \cdot & \phi_n^n \end{vmatrix} \qquad \text{(A-1.11)}$$

where $\tilde{\Phi}(t)$ satisfies $\frac{d\tilde{\Phi}}{dt}(t) = P(t)\,\tilde{\Phi}(t)$. The nonhomogeneous vector differential equation (A-1.9) then has the unique solution

$$\underline{\tilde{x}}(t) = \int_0^t \tilde{\Phi}(t)\tilde{\Phi}^{-1}(s)Q(s)\underline{\tilde{y}}(s)ds \quad . \qquad \text{(A-1.12)}$$

<u>An Alternate State Space Representation</u>

With the matrices P and Q as defined previously, the vector $\underline{\tilde{y}}(t)$ contains as elements the scalar y(t) and its (m-1) derivatives. It is frequently convenient to employ an alternate state space representation where the input vector contains only the scalar y(t) with none of its derivatives. For m n this alternate representation of equation (A-1.1) can be accomplished as follows. For generality, let m=n-1 and make the identifications

$$x_1 = x$$

$$\dot{x}_1 = x_2 - a_{n-1}x_1 + b_m y$$

$$\dot{x}_2 = x_3 - a_{n-2}x_1 + b_{m-1}y$$

$$\cdot$$

$$\cdot$$

$$\dot{x}_{n-1} = x_n - a_1 x_1 + b_1 y$$

$$\dot{x}_n = -a_o x_1 + b_o y \quad .$$

Now the vector equation equivalent to the scalar equation (A-1.1) is

$$\underline{\dot{x}} = A(t)\underline{x} + B(t)y \qquad \text{(A-1.13)}$$

$$\underline{x}(0) = 0 \quad ,$$

where

$$\underline{x} = \begin{vmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{vmatrix} \quad , \qquad x_1 = x$$

$$A = \begin{vmatrix} -a_{n-1} & 1 & o & \cdot & o \\ -a_{n-2} & o & 1 & \cdot & o \\ \cdot & \cdot & \cdot & \cdot & o \\ -a_1 & \cdot & \cdot & \cdot & 1 \\ -a_o & o & \cdot & \cdot & o \end{vmatrix} \quad , \quad B = \begin{vmatrix} b_m \\ b_{m-1} \\ \cdot \\ b_1 \\ b_o \end{vmatrix} \quad .$$

The a_ν and b_ν of equation (A-1.13) can be related to the p_ν and q_ν in equation (A-1.1) by rewriting equation (A-1.13) as

$$x^{(n)} = -\sum_{j=o}^{n-1-j}\sum_{k=o}^{n-1-k} \frac{(n-1-j)!}{k!(n-1-j-k)!} a_{n-1-j}^{(n-1-j-k)} x^{(k)} +$$

$$\sum_{j=o}^{m-1-j}\sum_{k=o}^{m-1-k} \frac{(m-1-j)!}{k!(m-1-j-k)!} b_{m-1-j}^{(m-1-j-k)} y^{(k)} \quad .$$

Then by comparing equation (A-1.14) to equation (A-1.1) there results

$$p_k = \sum_{j=o}^{n-1-k} \frac{(n-1-j)!}{k!(m-1-j-k)!} a_{n-1-j}^{(n-1-j-k)}$$

and

$$q_k = \sum_{j=o}^{m-1-k} \frac{(m-1-j)!}{k!(m-1-j-k)!} b_{m-1-j}^{(m-1-j-k)} \quad . \qquad \text{(A-1.15)}$$

The equations (A-1.15) can be solved sequentially to relate the p_k to a_k and q_k to b_k.

If $\phi_1, \phi_2, \cdots, \phi_n$ are n linearly independent solutions to the homogeneous equation (A-1.4) then the fundamental matrix for the homogeneous vector differential equation

$$\underline{\dot{x}} = A(t)\ \underline{x}$$

is given by

$$\Phi(t) = \begin{vmatrix} \phi_1 & \phi_2 & \circ & \circ & \phi_n \\ \phi_{12} & \phi_{22} & \circ & \circ & \phi_{n2} \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \phi_{1n} & \phi_{2n} & \circ & \circ & \phi_{nn} \end{vmatrix} \qquad \text{(A-1.16)}$$

where

$$\phi_{ij} = \frac{d}{dt}\phi_{ij-1} + a_{n-j-1}\,\phi_i \quad .$$

The nonhomogeneous vector differential equation (A-1.13) then has the unique solution

$$\underline{x}(t) = \int_0^t \Phi(t)\,\Phi^{-1}(s)B(s)y(s)ds \quad . \qquad \text{(A-1.17)}$$

APPENDIX II

REPRESENTATION OF A RANDOM PROCESS

The purpose of this appendix is to show that a Gaussian random variable $\hat{w}(t)$ can be represented as the output of a linear filter excited by the stationary white noise process discussed in Chapter III. Mathematically, what is to be shown is that $\hat{w}(t)$ can be expressed as

$$\hat{w}(t) = \int_0^t G(t,s)\mathring{u}(s)ds \quad . \qquad \text{(A-2.1)}$$

where $\mathring{u}(t)$ is a stationary white noise process and $G(t,s)$ is a physically realizable network impulse response function. The impulse response function $G(t,s)$ is not required to be stationary so that in general the random variable $\hat{w}(t)$ is nonstationary.

As preliminary, let $\{\beta_K(t)\}$ be a sequence of nonrandom continuous functions, orthonormal on the real time interval $[0,t]$ and complete in L_2 space, so that

$$\int_0^t \beta_i(s)\ {}_j(s)ds = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad . \qquad \text{(A-2.2)}$$

Then, from the expansion and completeness theorem*, any f(t)

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*op. cit. (20), pg. 197.

with n continuous derivatives on [0,t] has the representation in terms of the β_κ,

$$f(t) = \sum_{\kappa=0}^{\infty} [\int_0^t f(s)\beta_\kappa(s)ds]\beta_\kappa(t) \qquad \text{(A-2.3)}$$

where the sum is uniformly convergent on [0,t].

Let $\{\alpha_j\}$ be a sequence of random variables satisfying

$$E[\alpha_i\alpha_j] = \delta_{ij} \quad . \qquad \text{(A-2.4)}$$

Then for any random variable, z(t), such that the Stieltjes integral

$$\int_0^t f(s)dz(s)$$

exists on [0,t] for f continuous, the orthogonal decomposition theorem* states that z(t) has the expansion

$$z(t) = \sum_{i=0}^{\infty} \lambda_i\alpha_i\beta_i(t) \qquad \text{(A-2.5)}$$

if, and only if

$$E[z(t)z(s)] = \sum_{i=0}^{\infty} \lambda_i^2\beta_i(t)\beta_i(s) \quad . \qquad \text{(A-2.6)}$$

In equations (A-2.5) and (A-2.6), equality means convergence in the quadratic mean and λ_i are nonrandom constants.

Let G'(t,s) be the impulse response of any stable network and let

$$y(t) = \int_0^t G'(t,s)\dot{u}(s)ds \quad . \qquad \text{(A-2.7)}$$

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*Loeve (24), pg. 576.

Then

$$E[y(t_1)y(t_2)] = r_{yy}(t_1,t_2) = \int_0^{\min(t_1,t_2)} G'(t_1,\omega)g'(t_2,\omega)d\omega \quad . \quad (A\text{-}2.8)$$

Now, define a random variable $\mathring{u}(t)$ by

$$\mathring{u} = \sum_{i=0}^{\infty} \lambda_i \alpha_i \beta_i(t) \quad . \qquad (A\text{-}2.9)$$

Then let

$$\hat{y}(t) = \int_0^t G'(t,s)\dot{\hat{u}}(s)ds \qquad (A\text{-}2.10)$$

and then

$$r_{\hat{y}\hat{y}} = \int_0^{t_1} \int_0^{t_2} G'(t_1,s)G'(t_2,\omega) \sum_{i=0}^{\infty} \lambda_i^2 \beta_i(s)\beta_i(\omega)d\omega ds \qquad (A\text{-}2.11)$$

$$= \int_0^{t_1} G'(t_1,s) \sum_{i=0}^{\infty} [\int_0^{t_2} G'(t_2,\omega)\lambda_i\beta_i(\omega)d\omega]\lambda_i\beta_i(s)ds \quad .$$

Then since $G'(t,s)$ has n continuous derivatives on $[0,t]$, it follows from the expansion and completeness theorem that equation (A-2.11) is equivalent to

$$r_{\hat{y}\hat{y}}(t_1,t_2) = \int_0^{\min(t_1,t_2)} G'(t_1,s)G'(t_2,s)ds = r_{yy}(t_1,t_2) \quad . \quad (A\text{-}2.12)$$

Thus, from equation (A-2.12), the random variable $\mathring{u}(t)$, defined by equation (A-2.9), has the properties of white noise so that the white noise process $\dot{u}(t)$ formally has the representation

$$\dot{u}(t) = \sum_{i=0}^{\infty} \lambda_i \alpha_i \beta_i(t) \quad . \qquad (A\text{-}2.13)$$

Now consider the Gaussian random variable $\hat{w}(t)$ with covariance function

$$r_{\hat{w}\hat{w}}(t_1,t_2) = E[\hat{w}(t_1)\hat{w}(t_2)] < \infty \quad , \qquad \text{(A-2.14)}$$

where t_1, t_2 belong to the interval $[0,t]$.

By Mercer's theorem*, $r_{\hat{w}\hat{w}}$ has the expansion

$$r_{\hat{w}\hat{w}}(t_1,t_2) = \sum_{n=o}^{\infty} \gamma_n^2 \beta_n(t_1)\beta_n(t_2) \qquad \text{(A-2.15)}$$

where equality means convergence in the quadratic mean. By the orthogonal decomposition theorem, $\hat{w}(t)$ then has the expansion

$$\hat{w}(t) = \sum_{n=o}^{\infty} \gamma_n^2 d_n \beta_n(t) \qquad \text{(A-2.16)}$$

where the d_n are mutually independent random numbers with $E[d_i^2] = 1$.

Define a realizable network impulse response function by

$$G(t,s) = \begin{cases} 0 & (t<s) \\ \sum_{k=o}^{\infty} \mu_k \beta_k(t)\beta_k(s) & (t\geq s) \end{cases} \qquad \text{(A-2.17)}$$

Then using equation (A-2.13)

$$\int_o^t G(t,s)\dot{u}(s)ds = \int_o^t G(t,s) \sum_{i=o}^{\infty} \lambda_i \alpha_i \beta_i(s)ds \qquad \text{(A-2.18)}$$

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*<u>op</u>.<u>cit</u>. (21), pg. 374.

and for $t \geq s$

$$\int_0^t G(t,s)\dot{u}(s)ds = \int_0^t \sum_{k=0}^{\infty} \mu_k \beta_k(t)\beta_k(s) \sum_{i=0}^{\infty} \lambda_i \alpha_i \beta_i(s)ds = \qquad \text{(A-2.19)}$$

$$\sum_{k=0}^{\infty} \mu_k \lambda_k \alpha_k \beta_k(t) \quad .$$

Then by choosing

$$\mu_k = \frac{\gamma_k d_k}{\lambda_k \alpha_k} \qquad \text{(A-2.20)}$$

and using equation (A-2.16), equations (A-2.19) and (A-2.20) yield

$$\hat{w}(t) = \int_0^t G(t,s)\dot{u}(s)ds \qquad \text{(A-2.21)}$$

which is the desired result.

APPENDIX III

SPECIFICATION OF THE MODEL FOR A RANDOM PROCESS

This appendix is devoted to presenting a method specifying the elements of the mathematical model for a random process represented by equation (3.2). In particular, the material presented here enables the elements of the matrices $A^{(w)}$ and $B(t)$ in equation (3.2) to be specified in terms of the covariance function of the given random process. The material presented here essentially follows the work of Levy (15).

It is known* that the function $G(t,s)$ defined by equation (A-2.17) is the Green's function for an appropriate n^{th} order linear differential equation. That is, $G(t,s)$ satisfies

$$L_t(G(t,s)) = 0 \qquad \text{(A-3.1)}$$

where L_t is defined by equation (A-1.2). Then a function $w(t)$ which satisfies

$$w(t) = \int_o^t G(t,s)\mathring{u}(s)ds \qquad \text{(A-3.2)}$$

is the solution to the equivalent differential equation

$$L_t w(t) = N_t \mathring{u}(t) \quad . \qquad \text{(A-3.3)}$$

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*<u>op.cit.</u> (20), pg. 199.

The covariance function for the random variable $w(t)$ defined by equation (A-3.2) is

$$r_{ww}(t_1,t_2) = \int_0^{\min(t_1,t_2)} G(t_1,s)G(t_2,s)ds \quad . \qquad \text{(A-3.4)}$$

Then from equation (A-3.1)

$$L_{t_1} r_{ww}(t_1,t_2) = 0 \qquad (t_1>t_2) \quad . \qquad \text{(A-3.5)}$$

In many cases the covariance defined by equation (A-3.4) is representable by a finite sum of the form

$$r_{ww}(t_1,t_2) = \sum_{i=1}^{n}\phi_i(t_1)\gamma_i(t_2) \qquad (t_1>t_2). \qquad \text{(A-3.6)}$$

Otherwise, Mercer's theorem guarantees that any bounded covariance function can be approximated to any required accuracy by such a sum.

From equations (A-3.5) and (A-3.6) it follows that the n ϕ_i are solutions of the homogeneous differential equation (A-3.5). If n is the smallest number for which the sum (A-3.6) can be written, then the n ϕ_i are linearly independent and constitute a fundamental set of solutions for equation (A-3.5). Using this fundamental set, the coefficients $p_i(t)$ of the operator L_t are determined by means of equation (A-1.5). The elements of the matrix $A^{(w)}$ can then be determined by means of equation (A-1.5). Then matrix $A^{(w)}$ from equation (3.9) is then completely specified.

It remains to specify the elements of the matrix $B(t)$ of equation (3.9). The remainder of this appendix is devoted to this task.

From equation (A-1.17), the vector differential equation has the solution

$$\underline{w}(t) = \int_0^t \Phi(t)\,\Phi^{-1}(s)B(s)\mathring{u}(s)ds \qquad \text{(A-3.7)}$$

where $\Phi(t)$ is the fundamental matrix defined by equation (A-1.16) and is completely specified by the $\phi_i(t)$. The covariance matrix for the random vector $\underline{w}(t)$ can be expressed as

$$R_{\underline{ww}}(t_1,t_2) = E[\underline{w}(t_1)\underline{w}(t_2)^T] = \int_0^t \Phi(t')\,\Phi^{-1}(s)B(s)B^T(s)\Phi^{T-1}(s)\Phi^T(t)ds \qquad \text{(A-3.8)}$$

where $t' = \max(t_1,t_2)$, $t = \min(t_1,t_2)$.

Identifying a matrix $D(t)$ by

$$D(t) = \int_0^t \Phi^{-1}(s)B(s)B^T(s)\Phi^{T-1}(s)ds \qquad \text{(A-3.9)}$$

equation (A-3.8) can be rewritten as

$$R_{\underline{ww}}(t_1,t_2) = \Phi(t')D(t)\Phi^T(t) \quad . \qquad \text{(A-3.10)}$$

The element in the first row and first column of $R_{\underline{ww}}$ is just r_{ww} so that equations (A-3.6) and (A-3.10) yield

$$\sum_{i=1}^{n} \phi_i(t_1)\gamma_i(t_2) = \sum_{i=1}^{n} \phi_i(t_1) \sum_{j=1}^{n} d_{ij}(t_2)\phi_j(t_2) \qquad \text{(A-3.11)}$$

$$(t_1 > t_2) \quad ,$$

where the d_{ij} are elements in D. Equation (A-3.11) is satisfied by

$$d_{ij} = 0 \quad i \neq j \tag{A-3.12}$$

$$d_{ii}(t_2) = \frac{\gamma_i(t_2)}{\phi_i(t_2)} \quad .$$

Then using equation (A-2.16) to determine $\Phi(t)$, the elements of $R_{\underline{ww}}$ are completely specified.

Now, let $R^*_{\underline{ww}}(t_1,t_2)$ be the extension of equation (A-3.8) when the sign of the difference (t_1-t_2) changes. Then

$$R^*_{\underline{ww}}(t_1,t_2) = \int_0^{t'} \Phi(t)\Phi^{-1}(s)B(s)B^T(s)\Phi^{T-1}(s)\Phi^T(t')ds \tag{A-3.13}$$

$$= R^T_{\underline{ww}}(t_2,t_1) \quad .$$

Following Levy, let $\Delta(t_1,t_2)$ denote the difference

$$\Delta(t_1,t_2) = R_{\underline{ww}} - R^*_{\underline{ww}} = -\int_t^{t'} \Phi(t_1)\Phi^{-1}(s)B(s)B^T(s)\Phi^{T-1}(s)\Phi^T(t_2)ds \quad . \tag{A-3.14}$$

From equation (A-3.14) with $t_1>t_2$

$$\left.\frac{\partial\Delta(t_1,t_2)}{\partial t_1}\right|_{t_1=t_2} = -B(t_1)B^T(t_1) \quad . \tag{A-3.15}$$

The diagonal elements of the matrix $B\,B^T$ are just b_m^2, b_{m-1}^2, $\cdots$, b_1^2, b_o^2 so that the elements b_i of the matrix B are determined by

$$b_{m-i+1} = \sqrt{|\dot{\delta}_{ii}|} \tag{A-3.16}$$

where the $\dot{\delta}_{ii}$ are the diagonal elements of

$$\left. \frac{\partial \Delta(t_1,t_2)}{\Delta t_1} \right|_{t_1 = t_2} .$$

Thus the matrix B is completely specified.

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